THE COMPUTATION OF ORBITS

THE COMPUTATION OF ORBITS

by

Paul Herget, Ph. D.

Professor of Astronomy, University of Cincinnati

and

Director of the Cincinnati Observatory

Published privately by the author

1948

Lithoprinted from copy supplied by author
by
Edwards Brothers, Inc.
Ann Arbor, Michigan, U.S.A.
1948

Dedicated to my wife

HARRIET

who has lovingly made my
every wish her command,
my every need her duty.

PREFACE

This volume has been developed from the author's lectures on orbit computation. It is designed to serve as a textbook for a course which may pursue the subject to varying degrees of intensity and with different emphases. It is especially intended to provide a large amount of computational work; and one computing laboratory session may be scheduled for every one or two lecture periods, as the teacher chooses. There is more material than may be readily covered in a one-year course. The student's only required preparation for this work is a course in calculus, but the more mathematical background he has, the better.

It is expected that not only students of Astronomy, but also many Mathematics students will find a compelling interest in a subject which offers so many illustrations of mathematical principles, and which can be mastered so securely by careful numerical examples. In order to maintain a nicely balanced parallel between the computations and the text, and in order to increase the interest to the student, the material has been arranged for presentation in a continually unfolding psychological order, instead of a strictly logical order. This also makes it possible to provide a nearly continuous flow of computational work for the laboratory sessions, no matter at what level the course is set.

The emphasis has been placed on minor planet orbits at the beginning, instead of comet orbits, for several reasons. In the first place, for the beginning student, it requires a minimum of theory, and from comparable observations, the solution usually is obtained more readily. After the student has gained a fair mastery of minor planet orbit determination, he may progress on to comet orbit work, which is of a slightly higher order of difficulty.

Secondly, there are more minor planet orbits to be computed than comet orbits, but the latter are mostly work for experts, because the results are needed as soon as possible. On the other hand, Astronomy might well collect a coterie of workers who are not now actively engaged in any research and who can not give their time freely to the demands of such work, but who could work out a minor planet ephemeris over a period of several weeks and have it ready for the observer at the next dark of the Moon. Then when the apparition ends, there is nearly a year to work out the elements and prediction for the next opposition.

It is tacitly assumed that all the computations will be performed with the aid of a modern, hand-operated, desk-model, calculating machine. All the formulas and precepts have been prepared accordingly. At the present time this is the most popular manner of performing numerical computations, and for the computation of individual orbits it is still the most efficient means. No attempt has been made to cater for those computers who, for one reason or another, persist in the use of logarithms only. For the appropriate rearrangement of the formulas, they are left to their own devices, depending upon their experience and proficiency. On the other extreme, it has been deemed advisable not to reduce each and every computation to the form of Cracovians, mainly because this often introduces an artificiality which is not compensated by any especial advantage.

It is within the realm of physical possibility to make a book such as this practically complete, but it is hardly worthwhile. The question of what should be included has been answered mainly on the basis that there should be a complete core around which all types of orbit work can be built. Occasionally one of the side branches has been developed in detail. In other cases a reference indicates the direction the student should pursue to develop some branch by himself. In no case should the mature student feel that the book is a sufficient crutch for him to lean securely upon. It is only an opening wedge and a guide to a much larger field of material and an expertness of technique that can be attained only by diligent pursuit. The serious reader should develop from the very beginning the habit of keeping a bibliography and abstracts of the references he reads, and also of working out his own collection of formulas and arrangement of the computing forms to his own best advantage.

Even the most optimistic and biased opinion must recognize that the demand for a text in such a specialized field as this will be very limited. To produce the book in the usual way would have made the cost practically prohibitive, especially to some of the younger students whom it may serve. The author has not forgotten his own days in the company with other impecunious students. He has therefore undertaken to meet this economic "immovable object" with an economic "irresistible force" by producing the book in its present form. All the copy for photolithography has been prepared by the author with his own hands. At first this was undertaken as a hobby, but it eventually became an onerous task. It has occupied every available moment of spare time for one and a half years. But in no other way could the book have been produced at a reasonable price.

Unfortunately, these circumstances have caused two important disadvantages. The first is in the proofreading. The author can not guarantee the typographical accuracy to the full extent that is possible by lithoprinting in ordinary cases. Even more serious and regrettable is the author's inability to guarantee the accuracy of every digit in the numerical examples. This has just not been possible, in spite of its importance. In case of an apparent error, the student will have to attempt a check computation, by some other formula if possible, and decide on the correct value by his own devices. Discordances in end-figure rounding need not be pursued too far; they are not nearly as serious as the fetish which some computers make of them. The author will appreciate receiving notice of all errors of any kind that are detected.

The second disadvantage is in the notation. There is not a limitless number of different characters available, and the concessions to typographical stringency are often glaring. On the other hand, the author has tried to retain generally adopted notations as much as possible and the original notations whenever presenting material for which references are cited. The duplication which is thus introduced is evidence that the problem is not new. For these reasons, no attempt has been made to give a complete glossary of notation. If the reader feels the need for one, he will probably find it best to prepare separate ones for each different phase of the work, and thus separate most of the duplications.

The excellent appearance of the printed text is due to the use of a standard I B M Electromatic Proportional Spacing typewriter. Most of the onerous work was due to the characters which were not on the typewriter. The tables at the end of the volume were typed on a special, card-controlled, electric typewriter at the Watson Scientific Computing Laboratory. The author is indebted to Dr. W. J. Eckert, director of the laboratory, for this courtesy, and to Miss Rebecca Jones for her careful workmanship and attention to all the details. The binding has been chosen not only to help reduce the cost, but also so that the book will lie open and flat on the desk or computing table when it is in use. This is a distinct advantage; the conventional stiff-spine binding is unsatisfactory in this regard. The cover is sufficiently durable for ordinary usage. The fine appearance of the book as a whole is due to the unflagging cooperation, excellent processing, and high standards of craftsmanship attained by Edwards Bros., the lithoprinters, to whom the author is extremely grateful.

Paul Herget

Cincinnati Observatory
August, 1948

TABLE OF CONTENTS

INTRODUCTION

"Αγε νῦν, διαλύεσθε τάδε.

The prediction of the motions of the heavenly bodies was one of the earliest problems in Astronomy. At first these predictions were based on empirical rules; then they were based on the Ptolemaic system of epicycles, but as more and more observations accumulated this system was found to be unsatisfactory. Later the motions of the planets were based on the laws which Kepler discovered through his analysis of Tycho Brahe's observations of Mars. But then followed the invention of the telescope, which provided much more precise observational data than before and placed a correspondingly greater strain upon any theory which attempted to represent the motions of heavenly bodies. Notwithstanding, Newton's laws of motion and gravitation provided a basis for bringing the observed and the theoretically determined positions into more satisfactory agreement than ever before. After Halley's application of Newton's laws to the observations of twenty four comets, it became possible, for the first time, to predict the paths of the comets across the sky. During the eighteenth century, numerous mathematical investigations appeared which attempted to find the best means of applying the Newtonian laws. For determining the orbits of comets, Olbers' method has proved most practical. The simplest method in theory is that of La Place. The dawn of the nineteenth century heralded the first of the discoveries of minor planets, and this stimulus led Gauss, almost immediately, to the invention of his unsurpassed method of determining preliminary orbits. Since then modifications in methods have been presented with the view to facilitating the numerical work, the most recent being designed to take the fullest advantage of modern calculating machines.

In this volume we shall develop from fundamental principles several of the practical methods of computing the orbit and ephemeris of a newly discovered object in the solar system, and also means of subsequently improving these results. All the work is based upon the assumption that the object obeys Newton's laws of motion and gravitation. That Nature does not function strictly in accordance with these laws has been shown by the discordance between the observed and the theoretically predicted advance of the perihelion of Mercury, but for all other cases (except perhaps the Moon) the assumption is a sufficiently close approximation for most practical purposes. These laws may be stated as follows:

> A particle will continue in a state of rest or of uniform motion in a straight line unless acted upon by some force. (Law of Inertia)
>
> The action of a force upon a particle produces an acceleration which is proportional to the force and in the same direction, and inversely proportional to the mass of the particle. ($F = m\,a$)
>
> For every acting force there is an oppositely directed force of equal magnitude. (Action and Reaction)
>
> Every particle of matter attracts every other particle with a force that is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them. (Universal Gravitation)

It can be shown that when the material in a body is distributed in homogeneous concentric layers the total effect upon an external body is the same as if all the mass were acting from a point at the center of the body.

Our problem will be solved by substituting into the equation given by the second law the force of gravitation given by the last law, thus providing an equation for the acceleration, or, in other words, three second order differential equations for the three coordinates of the object in

space. The solution of these equations is a space curve which contains six constants of integration. These six quantities must be so determined that the positions computed from the solution agree with the positions that are actually observed on the sky. Thus we see that there are two different kinds of conditions which the solution must satisfy before the work is complete. The former are known as the dynamical conditions and they insure that the motion of the object from one point to another shall be in accordance with Newton's laws. The latter are known as the geometrical conditions and they insure that the motion shall be in accordance with the observations that are available.

The method of La Place is based upon a Taylor's series expansion about some instant of time and solves the differential equations from their numerical values. The method of Gauss is based upon an analytical solution of the differential equations and solves for the numerical values of the constants of integration. In the method of La Place the formulas may be arranged in such a way that the dynamical conditions are always satisfied and the object of the solution is to obtain such values of the unknowns as will satisfy the geometrical conditions. On the other hand, the formulas may be arranged so that the geometrical conditions are always satisfied and it remains to find such values as will satisfy the dynamical conditions. Similarly the solution by the method of Gauss may be attacked by either of these two approaches. Nearly all practical methods of orbit computation may be classified as belonging to one of these four cases or some combination of them. The reader will find a more extensive description of this subject in a paper by Woolard, The Calculation of Planetary Motions, in the National Mathematics Magazine, January, 1940.

It is assumed that the reader is familiar with the differential and integral calculus, but it is probably not reasonable to assume that he is equally familiar with the calculus of finite differences. Since this subject is of considerable importance to the present work (as indeed it is to all numerical work) a separate chapter has been devoted to a treatment of the basic portions of the calculus of finite differences with equal intervals of the argument. These derivations presume a knowledge only of the Taylor's series expansion of a function, a topic encountered in elementary calculus. To one who has had experience only with literal treatments of mathematical ideas, as the student has probably had in his courses in algebra and calculus, the process of obtaining results by these numerical methods may appear to rest upon some mystifying, "rabbit-out-of-the-hat" trick. It is recognized that this effect may even be aggravated by the present use of symbolic operators, but they bring to the developments such elegance, and are in themselves so powerful a method that the student will be well repaid for the extra effort required to master them. He will, however, soon discover for himself that it is only the resulting formulas which are essential to the subsequent work; in fact, this chapter may be deemed to belong more properly in an appendix than amongst the introductory topics. But this is a topic of real intrinsic importance and the author has tried to give it a deserving presentation. It is unfortunate that so many mathematicians hold the view that these numerical methods are of a lower caste (to some, the "untouchables"), when actually they are able to deal equally well with cases which are too complicated to be solved by the ordinary methods of analysis.

For the benefit of those students who have the mathematical preparation but are not familiar with the elements of spherical astronomy, the necessary fundamentals have been presented in their geometrical aspects, but merely to the extent that they are needed to solve the main problem as it exists in practice. Vector analysis has been introduced after a brief review, both to simplify the developments and to aid in visualizing the situation in the problem at hand. The elementary notions of position, velocity, and normal vectors help to visualize the geometrical relationships associated with the orbital motion, and it will often be an aid to make an isometric sketch of the vectors in space. It is one of the attractive attributes of this work that the results may be visualized at each stage, and every formula and operation has its actual physical counterpart in the problem. This makes for a better understanding of the work, greater interest, and often leads to the detection of accidental errors when the computer notices that the results are unreasonable or even absurd.

Finally, a word of caution about the numerical computations. Nothing is so treacherous in this work as a minus sign, and no amount of forewarning will insure completely against its pitfalls. Since we shall be dealing with positions in space which may lie in any direction from the origin, many of the quantities are as likely to be negative as positive, and the computer must develop a consciousness of sign at all times. Copying results from the machine to the computing sheet is another prolific source of error, as well as being very time consuming. For this reason it is best

to arrange the formulas, whenever possible, into a sequence of calculations which permits the accumulation of several products in the product dials with, perhaps, a final division, or else the transfer of a quantity from the product or quotient dials to the keyboard for another operation, all without requiring intervening recording. It will sometimes be an aid to keep these formulas free of minus signs by attaching the necessary signs to some of the factors. In this way the sign of a product depends only upon the visible signs of the two factors which are read from the computing sheet and not upon still a third sign which must be borne in mind. In dealing with mixed signs, the computer should invariably enter negative products into the product dials negatively and positive quantities positively. The negative results then appear as complement numbers on the machine, but one soon develops an ability to convert these mentally to the true figures almost as rapidly as they can be written. One is easily tempted to estimate in advance the sign of the result and then reverse all the signs as the products are entered into the machine in anticipation of a negative result. This practice tends to lead to some confusion and perhaps the oversight of a sign. On the other hand, when a fixed habit is invariably practiced, it almost seems as if ones subconscious produces warnings when errors in sign are about to be committed.

Several other habits which contribute to maintaining accuracy in the work may be mentioned for the beginner. A computing form and precepts for its use should be prepared in advance of the actual performance of the computations. It is at this point that the computer should guarantee to his own satisfaction his understanding and mastery of the problem to be solved, and he should arrive at a concept of the numerical results which are to be expected. The computing itself is a mechanical process and it should be performed in a routine fashion, following the computing form by rule of thumb. The operation of the calculating machine is an important skill, and bears about the same relationship to this work that the skill of handwriting does to our everyday activities. It is most convenient to operate the machine with the hand which one does not use for writing. Modern machines may be operated with either hand with equal facility; it is simply a problem in habit formation. This eliminates shifting the pencil (or pen!) to and fro and permits it to be used as a pointer when referring to previously recorded quantities. The transfer of a number from the product dials to the keyboard should be checked by subtracting the setting from the product counter (adding in case the result was a complement) to reduce it to zeros or nines before the machine is cleared for the next operation. A minor though important detail in division may be mentioned, namely, that of comparing half the divisor with the remainder to determine whether the quotient needs to have one more unit added to the last place for proper rounding. When an error has been detected, it is not sufficient merely to correct it; try to ascertain how it was made and correct the bad habit as well. Skill and accomplishment in this task of charting celestial objects provides a thrill of satisfaction that has few equals, and careful work is fully rewarded when the computer makes the final test by comparing his solution with the observations and the laurels of success are bestowed in the form of small residuals.

CHAPTER 1

THE CALCULUS OF FINITE DIFFERENCES

Οὗτος δ' αὐτοῖς ἀριθμοῖς τὴν κρατήσει γῆν.
— Shakespeare

Continuous functions which are too refractory to be treated by the analytical procedures of the differential and integral calculus may, nevertheless, be handled by the numerical methods of the calculus of finite differences. In practical applications, functions are often defined only by their numerical values, and no other type of treatment is possible. In other cases, such as those with which we shall be concerned, the needed results may be obtained much more readily and with much less work by the numerical methods. The theoretical development of the calculus of finite differences may be founded entirely upon Taylor's series. It requires simply that over the range of the argument concerned the function shall be continuous and have continuous derivatives of all orders.

In this work the representation of a function by means of a formula or an algebraic expression is replaced by a table of numerical values of the function corresponding to a succession of values of the argument. Only those cases in which the numerical values of the function are given at equal intervals of the argument will be treated in this chapter. Such tables are usually arranged in vertical columns, with increasing values of the argument running down the extreme left hand column. Let us adopt the following notation to represent the individual quantities in such a table, their differences, and summations.

Argument	2nd Sum.	1st Sum.	Fn.	1st Diff.	2nd Diff.	3rd Diff.	
$t_i - h$	${}^{II}f_{i-1}$		f_{i-1}		Δ^{II}_{i-1}		
		${}^{I}f_{i-1/2}$		$\Delta^{I}_{i-1/2}$		$\Delta^{III}_{i-1/2}$	
t_i	${}^{II}f_i$		f_i		Δ^{II}_i		
		${}^{I}f_{i+1/2}$		$\Delta^{I}_{i+1/2}$		$\Delta^{III}_{i+1/2}$	((1,1))
$t_i + h$	${}^{II}f_{i+1}$		f_{i+1}		Δ^{II}_{i+1}		
		${}^{I}f_{i+3/2}$		$\Delta^{I}_{i+3/2}$		$\Delta^{III}_{i+3/2}$	
$t_i + 2h$	${}^{II}f_{i+2}$		f_{i+2}		Δ^{II}_{i+2}		

Each quantity in the table is the sum of the quantity directly above it plus the quantity a half line above and in the column to the right. The uniformity and simplicity of the notation greatly aid the beginner in gaining familiarity with it. The vertical position of any quantity is indicated by its subscript. The differences of the function are all indicated by Δ's, and the order of the difference by the superscript. Similarly, quantities which correspond to the inverse of a difference and which are on the left side of the function column have their superscripts indicated on the upper left side of the f's, e.g. ${}^{II}f$ and ${}^{I}f$. Later on we shall need to insert values in the blank spaces "on the line" in the odd difference columns or "on the half line" in the even difference columns; these are obtained simply by taking the mean of the quantities a half line above and a half line below the space to be filled. Such values are usually enclosed in parentheses or written in some distinctive color.

We may notice, in passing, that a difference of any order is expressible directly in terms of the functions, and when this is done the functions are combined with coefficients which are the binomial numbers corresponding to the order of the difference and taken with alternating signs, e.g. $\Delta^{IV}_0 = f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}$. Also the presence of an error in some one value of the function

will cause the error to appear in the successive difference columns with coefficients which also are the binomial numbers corresponding to the order of the difference and taken with alternating signs, e.g. an error e in f_i will be attached to the third differences as follows:

$$\Delta^{iii}_{i-3/2} + e,\ \Delta^{iii}_{i-1/2} - 3e,\ \Delta^{iii}_{i+1/2} + 3e,\ \Delta^{iii}_{i+3/2} - e.$$

The reader may test this by deliberately introducing an error into a table of, say, the cubes of the integers. This property of differences provides a simple, yet powerful method of checking any computations which have been made at small, equal intervals of some parameter, and it is used a great deal in practice.

The relationships between the infinitesimal calculus and the calculus of finite differences may be illustrated by a simple example. Write

$$f(t) = f(t_i + h) = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \ldots,$$

where we see by comparison with a Taylor's series that $a_1 = \frac{df(t_i)}{dt}$, $a_2 = \frac{1}{2}\frac{d^2f(t_i)}{dt^2}$, etc. If we now substitute integral values for h, we have

$$\begin{aligned}
f(t_i - 2) &= a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 - 32a_5 + \ldots \\
f(t_i - 1) &= a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \ldots \\
f(t_i) &= a_0 \\
f(t_i + 1) &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \ldots \\
f(t_i + 2) &= a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 + 32a_5 + \ldots
\end{aligned}$$

Then

$$\begin{aligned}
\Delta^{i}_{i-1/2} &= f(t_i) - f(t_i - 1) = a_1 - a_2 + a_3 - a_4 + \ldots \\
\Delta^{i}_{i+1/2} &= f(t_i + 1) - f(t_i) = a_1 + a_2 + a_3 + a_4 + \ldots \\
\Delta^{i}_{i} &= \tfrac{1}{2}(\Delta^{i}_{i+1/2} + \Delta^{i}_{i-1/2}) = a_1 + a_3 + \ldots
\end{aligned}$$

Similarly, we may obtain $\Delta^{iii}_i = 6a_3 + 30a_5 + \ldots$. Then by eliminating a_3, we have

$$a_1 = \frac{df(t_i)}{dt} = \Delta^{i}_{i} - \Delta^{iii}_{i}/6 + \ldots,$$

which means that we are able to evaluate the first derivative of a function to a certain degree of accuracy from the numerical quantities in the table of differences. This formula is valid only for those values of the argument at which the function is evaluated in the table, not at intermediate values of the argument. We shall see later how this restriction can be removed.

Now let us examine the application of this formula to a portion of the table of t^4. We know from differential calculus that the result should be $4t^3$. We see from the adjoining table that at the values $t_i = \begin{cases}5\\6\end{cases}$, we have $\Delta^{i}_{i} = \begin{cases}520\\888\end{cases}$, $\Delta^{iii}_{i} = \begin{cases}120\\144\end{cases}$, and $\Delta^{i}_{i} - \Delta^{iii}_{i}/6 = \begin{cases}500\\864\end{cases}$.

t	Fn.	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.
4	256		194		24
		369		108	
5	625	(520)	302	(120)	24
		671		132	
6	1296	(888)	434	(144)	24
		1105		156	
7	2401		590		24

Both of these values check exactly with $4t^3$, due to the fact that we have chosen a function whose higher order differences are exactly zero, and therefore the neglected higher order difference terms in the formula have been forced to vanish. In general, this would not be the case, as the student may verify by testing a table of t^5.

If we undertook to extend this formula to include the effects of higher order differences, or to derive other formulas for derivatives of higher order or for integrals by this same "hammer

and tongs" method, the developments would be long and tedious. The results may, however, be obtained by simple, elegant methods if we employ symbolic operators. The following developments are based upon a short paper by G. W. Hill, Collected Mathematical Works, vol. 1, p. 181-184.

Define the symbolic operator D, operating upon the function f_i, in such a way that

$$D\{f_i\} = \lim_{\Delta t \to 0} \frac{f(t_i + \Delta t) - f(t_i - \Delta t)}{2\,\Delta t} \qquad ((1,2))$$

This is equivalent to the usual definition of the derivative, in fact, $D = \frac{d}{dt}$, but it is put in this form for the benefit of comparison with $\blacktriangle$ below. The student may notice that in this definition the chord joining the points $f(t_i + \Delta t)$ and $f(t_i - \Delta t)$ assumes positions which are nearly parallel to each other as it approaches the limiting position of the tangent. This is intuitively more appealing than the definition usually given in beginning calculus texts, in which the point $f(t_i - \Delta t)$ is replaced by $f(t_i)$ and the chord rotates about this point. Successive differentiations are denoted by raising D to successive powers, e.g. $D^2 = \frac{d^2}{dt^2}$, etc.

Define another symbolic operator $\blacktriangle$ operating upon the numerical values of the tabulated function f_i in such a way that

$$\blacktriangle\{f_i\} = \tfrac{1}{2}(f_{i+1} - f_{i-1}). \qquad ((1,3))$$

In the calculus of finite differences there is no infinitesimal to approach zero as in ((1,2)), in fact, Δt is now replaced by the interval of the argument, h, which is fixed. The operator $\blacktriangle$ has the effect of producing the mean first difference in the table on the line with f_i, for

$$\tfrac{1}{2}(f_{i+1} - f_{i-1}) = \tfrac{1}{2}[(f_{i+1} - f_i) + (f_i - f_{i-1})] = \tfrac{1}{2}(\Delta^{i}_{i+1/2} + \Delta^{i}_{i-1/2}) = \Delta^{i}_{i}.$$

Also define another operator Δ^2 operating upon the same tabulated function in such a way that

$$\Delta^2\{f_i\} = f_{i+1} - 2f_i + f_{i-1}. \qquad ((1,4))$$

This operator produces the second difference in the table on the line with f_i.

Now $\blacktriangle\{\blacktriangle\{f_i\}\} = \blacktriangle\{\tfrac{1}{2}(f_{i+1} - f_{i-1})\} = \tfrac{1}{2}[\tfrac{1}{2}(f_{i+2} - f_i) - \tfrac{1}{2}(f_i - f_{i-2})] = \frac{1}{4}(f_{i+2} - 2f_i + f_{i-2}) \neq \Delta^2\{f_i\}$.

In like manner, the student may verify the other relationships which follow:

$$\blacktriangle\{\blacktriangle\{f_i\}\} \neq \Delta^2\{f_i\}, \quad \Delta^2\{\Delta^2\{f_i\}\} = \Delta^4\{f_i\} = \Delta^{iv}_i, \quad \blacktriangle\{\Delta^{2k}\{f_i\}\} = \Delta_i^{(2k+1)}. \qquad ((1,5))$$

The inequality states that the mean first differences of the mean first differences of a function are not the same as its second differences, and therefore the square of the first operator is not equivalent to the second operator. The first equation states that the second differences of the second differences are the fourth differences. In fact, the even exponents of the second operator may be added as in ordinary algebra; the resulting quantity is always the even order difference on the same line and in the column indicated by the exponent. The second equation states that the quantity "on the line" in any odd difference column is the mean first difference of the quantities in the even difference column of one order lower. The reader will perceive that the symbolic operators Δ^{2k} and $\blacktriangle\Delta^{2k}$ are simply shorthand representations to signify the quantities "on the line" in the table of differences. Later we shall obtain expressions for the quantities "on the half line".

To ascertain the laws which govern the algebra of these two symbolic operators and the derivatives of the function, we shall use as intermediary the symbolic form of Taylor's series. Let e denote the exponential (in this chapter only) and write

$$e^{hD} = 1 + h\frac{d}{dt} + \frac{h^2 d^2}{2!\,dt^2} + \frac{h^3 d^3}{3!\,dt^3} + \cdots.$$

Choose for h the constant value of the interval of the argument in the table; then

$$e^{hD}\{f_i\} = f(t_i + h) = f_{i+1}, \text{ and } e^{-hD}\{f_i\} = f_{i-1}.$$

If we substitute these expressions into ((1,3)) and ((1,4)) we obtain the following equations connecting the symbolic operators:

$$\blacktriangle = \tfrac{1}{2}(e^{hD} - e^{-hD}) = \tfrac{1}{2}(e^{hD/2} + e^{-hD/2})(e^{hD/2} - e^{-hD/2})$$

$$\Delta^2 = e^{hD} - 2 + e^{-hD} = (e^{hD/2} - e^{-hD/2})^2, \quad 4 + \Delta^2 = e^{hD} + 2 + e^{-hD} = (e^{hD/2} + e^{-hD/2})^2$$

$$\sqrt{\Delta^2} = e^{hD/2} - e^{-hD/2} \qquad e^{hD/2} = \tfrac{1}{2}\sqrt{\Delta^2} + \sqrt{1 + \Delta^2/4}$$

$$\sqrt{4 + \Delta^2} = e^{hD/2} + e^{-hD/2} \qquad \bar{\Delta} = \sqrt{\Delta^2}\sqrt{1 + \Delta^2/4} \tag{(1,6)}$$

$$hD/2 = \ln\left(\tfrac{1}{2}\sqrt{\Delta^2} + \sqrt{1 + \Delta^2/4}\right).$$

Let us now examine the meaning and validity of these formal operations. We have obtained, in effect, a series relationship between $h\dfrac{df}{dt}$ and the various orders of finite differences. If we expand the natural logarithm as a power series, it gives hD as an odd series in powers of $\sqrt{\Delta^2}$. We have, however, a definition only of even powers of Δ from ((1,4)). Therefore let us take out a factor $\sqrt{\Delta^2}$, leaving the remaining factor an even series, and eliminate this undefined term by means of $\sqrt{\Delta^2} = \bar{\Delta}/\sqrt{1 + \Delta^2/4}$ from ((1,6)). Then if we wish to have an odd power of Δ become a symbolic operator which produces the quantity "on the line" in the corresponding odd order difference column of the numerical table, it must be defined in accordance with the second equation of ((1,5)), namely $\Delta^{2k+1} = \bar{\Delta}\Delta^{2k}$. If we make this substitution and compare the resulting expression with our original expansion, we see that it is still the formal expansion for the natural logarithm, now in odd powers of Δ, but also with a new factor $\sqrt{1 + \Delta^2/4}$ in the denominator. The purpose of introducing the notation $\bar{\Delta}$ has been to keep a clear distinction among $\bar{\Delta}$, $\sqrt{\Delta^2}$, and Δ.

If we consider the square or any even power of (hD), the expansion will contain only even powers of Δ, and no difficulty will be encountered. Also if we replace the natural logarithm by its equivalent expression as an integral, we shall be able to obtain the coefficients of the series the more readily. Thus we have, in general,

$$(hD)^{2k} = \left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k}, \qquad (hD)^{2k+1} = \frac{1}{\sqrt{1 + \Delta^2/4}}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k+1}. \tag{(1,7)}$$

These results indicate that the derivatives of all orders may be expressed as power series of the differences. It is not necessary to expand these expressions in order to obtain the values of the coefficients; they may be obtained by developing a recursion formula by means of the method of undetermined coefficients on the basis of the differential relations which exist. It is evident from ((1,7)) that

$$\frac{d}{d\Delta}(hD)^{2k} = \frac{2k}{\sqrt{1 + \Delta^2/4}}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k-1} = 2k\,(hD)^{2k-1}$$

or

$$(hD)^{2k} = 2k\int(hD)^{2k-1}d\Delta.$$

Also

$$\frac{d}{d\Delta}(hD)^{2k+1} = \frac{(2k+1)}{1 + \Delta^2/4}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k} - \frac{\Delta}{4(1 + \Delta^2/4)^{3/2}}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k+1}$$

or

$$(1 + \Delta^2/4)\frac{d}{d\Delta}(hD)^{2k+1} + \frac{\Delta}{4}(hD)^{2k+1} - (2k+1)(hD)^{2k} = 0.$$

Let $\displaystyle (hD)^{2k+1} = \sum_{j=0}^{\infty} A_j^{(2k+1)}\Delta^j;$ then $\displaystyle \frac{d}{d\Delta}(hD)^{2k+1} = \sum_{j=1}^{\infty} jA_j^{(2k+1)}\Delta^{j-1}$, and

$$(1 + \Delta^2/4)\sum jA_j^{(2k+1)}\Delta^{j-1} + \frac{\Delta}{4}\sum A_j^{(2k+1)}\Delta^j - (2k+1)\sum A_j^{(2k)}\Delta^j = 0.$$

Equate the coefficient of Δ^{j+1} to zero and transpose.

$$A_{j+2}^{(2k+1)} = \frac{2k+1}{j+2}A_{j+1}^{(2k)} - \frac{j+1}{4(j+2)}A_j^{(2k+1)} \tag{(1,8)}$$

To find the leading coefficients, we have $A_j^{(0)} = 0$, except $A_0^{(0)} = 1$, and

$$(hD) = \frac{1}{\sqrt{1 + \Delta^2/4}}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right) = \left(1 - \frac{\Delta^2}{8} + \ldots\right)\left(\Delta - \frac{\Delta^3}{24} + \ldots\right) = \Delta - \frac{\Delta^3}{6} + \ldots,$$

therefore $A_1^{(1)} = 1$, $A_2^{(1)} = 0$, and as a check, $A_3^{(1)} = -\frac{1}{6}$.

By repeated application of the recursion formula ((1,8)) when passing from $(hD)^{2k}$ to $(hD)^{2k+1}$ or by integration when passing from $(hD)^{2k-1}$ to $(hD)^{2k}$ we obtain all the following formulas:

$$hD = \Delta - \frac{1}{3}\frac{\Delta^3}{2} + \frac{1}{3}\frac{2}{5}\frac{\Delta^5}{2^2} - \frac{1}{3}\frac{2}{5}\frac{3}{7}\frac{\Delta^7}{2^3} + \frac{1}{3}\frac{2}{5}\frac{3}{7}\frac{4}{9}\frac{\Delta^9}{2^4} - \cdots$$

$$(hD)^2 = \Delta^2 - \frac{1}{3}\frac{1}{2}\frac{\Delta^4}{2} + \frac{1}{3}\frac{2}{5}\frac{1}{3}\frac{\Delta^6}{2^2} - \frac{1}{3}\frac{2}{5}\frac{3}{7}\frac{1}{4}\frac{\Delta^8}{2^3} + \cdots$$

$$(hD)^3 = \Delta^3 - \frac{1}{4}\Delta^5 + \frac{7}{120}\Delta^7 - \frac{41}{3024}\Delta^9 + \cdots$$

$$(hD)^4 = \Delta^4 - \frac{1}{6}\Delta^6 + \frac{7}{240}\Delta^8 - \frac{41}{7560}\Delta^{10} + \cdots \qquad ((1,9))$$

$$(hD)^5 = \Delta^5 - \frac{1}{5}\Delta^7 + \frac{13}{144}\Delta^9 - \cdots$$

$$(hD)^6 = \Delta^6 - \frac{1}{4}\Delta^8 + \frac{13}{240}\Delta^{10} - \cdots$$

These results and others given below may also be found in Oppolzer's Lehrbuch zur Bahnbestimmungen, vol. 2, and in the British Nautical Almanac for 1937.

The negative powers of D will correspond to antiderivatives or successive integrations of the function. The coefficients of the series for $(hD)^{-2}$ may be obtained from the reciprocal of the series for $(hD)^2$ by long division, and then the recursion formula (with k = - 1) will give the series for $(hD)^{-1}$.

$$(hD)^{-2} = \Delta^{-2} + \frac{1}{12} - \frac{1}{240}\Delta^2 + \frac{31}{60480}\Delta^4 - \frac{289}{3628800}\Delta^6 + \frac{317}{22809600}\Delta^8 - \cdots \qquad ((1,10))$$

$$(hD)^{-1} = \Delta^{-1} - \frac{1}{12}\Delta + \frac{11}{720}\Delta^3 - \frac{191}{60480}\Delta^5 + \frac{2497}{3628800}\Delta^7 - \frac{14797}{95800320}\Delta^9 + \cdots \qquad ((1,11))$$

These formulas enable us to evaluate (but only at the tabular values of the argument) the derivatives and the integrals of any continuous function which is defined by its numerical values at equal intervals of the argument. Two examples will be given to illustrate their application. First, consider the following table:

Argument	1st Sum.	Fn.	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.	5th Diff.	6th Diff.
-0.1		-0.00001		-30		-120		0
	0.00000		1		30		120	
0.0	(0.00000)	0.00000	(1)	0	(30)	0	(120)	0
	0.00000		1		30		120	
0.1		0.00001		30		120		0
	0.00001		31		150		120	
0.2		0.00032		180		240		0
	0.00033		211		390		120	
0.3		0.00243		570		360		0
	0.00276		781		750		120	
0.4		0.01024		1320		480		0
	0.01300		2101		1230		120	
0.5	(0.028625)	0.03125	(3376)	2550	(1530)	600	(120)	0
	0.04425		4651		1830		120	
0.6		0.07776		4380		720		0

In this table the interval is 0.1 and the function is t^5. If, for comparison, we first obtain analytically the expressions for the single integral and the first, second, and third derivatives, and evaluate these for t = 0.5, we obtain 1/384, 5/16, 5/2, and 15, respectively. From ((1,11)), ((1,9)), and the values of the differences in the above table on the line with t = 0.5, we obtain:

$$(hD)^3\{f_i\} = 0.001\frac{d^3f}{dt^3} = 0.01530 - (0.00120)/4 = 0.01500$$

$$(hD)^2\{f_i\} = 0.01\frac{d^2f}{dt^2} = 0.02550 - (0.00600)/12 = 0.02500$$

$$(hD)\{f_i\} = 0.1\frac{df}{dt} = 0.03376 - (0.01530)/6 + (0.00120)/30 = 0.03125$$

$$(hD)^{-1}\{f_i\} = 10\int f\,dt = 0.028625 - (0.03376)/12 + (0.01530)\,11/720 - (0.00120)\,191/60480 = 10/384 - 10/252{,}000{,}000.$$

$$0.00000 - (0.00001)/12 + (0.00030)\,11/720 - (0.00120)\,191/60480 = -\,10/252{,}000{,}000.$$

The values for the three derivatives are in exact agreement with those we obtained above, but the value for the integral requires further explanation. The quantity (0.00000) which was placed in the 1st Sum. column opposite $t = 0$ corresponds to the usual arbitrary constant of integration. Its correct value to five decimal places is zero if the integral is to vanish with t, but when its value is taken to be exactly zero it does not cause the integral to vanish exactly, as can be seen from the last line of figures above. Since the integral is too small by this amount at the origin, it remains too small by this amount throughout. If all the computed values of the integral are increased by this amount, the agreement is then exact. Similarly, by the proper adjustment of the arbitrary quantity in the 1st Sum. column, the integral may be caused to assume any desired value at some selected point.

Second, consider the differential equation $\frac{d^2x}{dt^2} = -x$, where the initial conditions are $x = 2$ and $\frac{dx}{dt} = 0$ at $t = 0$. Here our problem is to obtain x by means of the double integration of a function which is simply $f(x,t) = -x$. Since the values in the function column cannot be calculated until the integral is known, it might appear that we have reached an impasse. In practice, it is necessary to proceed by extrapolating the solution one interval at a time. It is also advantageous to include the factor h^2 in the computed values of the function so that it does not need to be taken into account later every time the numerical value of the integral is computed. Let the interval of the table be 0.1, then the value to be computed for the function column is $-0.01x$.

Once the table has been started, the procedure is as follows: calculate x from ((1,10)):

$$x = D^{-2}\{f_i\} = {}^{II}f_i + \frac{1}{12}f_i - \frac{1}{240}\Delta_i^{II} + \ldots$$

where ${}^{II}f_i$ is the last known quantity in the 2nd Sum. column, f_i must be estimated from the run of the first differences, and Δ_i^{II} is neglected. Then to compute the function f_i, simply point off two decimal places in the value of x just derived and change the sign. As a check, recompute x with the same formula, but this time using for f_i the value just derived, and for Δ_i^{II} a value estimated from the run of the differences. Also recompute the function, if necessary. If this check agrees, the whole table may be extended one line farther down, and the process is repeated.

To get the table started, it is necessary to estimate the values of the function not from the run of the differences (of which there are as yet none) but from the initial conditions; and then the integral formulas ((1,10)) and ((1,11)) are used not to evaluate the integrals, since these are given by the initial conditions, but to determine the initial values in the summation columns so as to satisfy the given initial conditions. Thus at $t = 0$, the value of the function is -0.0200, and since the velocity of x is zero at this point we shall use the same value as a first approximation at the two neighboring points. Then by ((1,10)), $2.0 = {}^{II}f_0 + (-0.0200)/12 + \ldots$, and by ((1,11)), ${}^{I}f_0 = 0$.

t	2nd Sum.	1st Sum.	Fn.	1st Diff.
-0.1	1.9917		-0.0200	
		+0.0100		0
0.0	2.0017		-0.0200	
		-0.0100		0
0.1	1.9917		-0.0200	

If we insert these values into our table, it then appears as shown at the right. Next we use ((1,10)) to evaluate the x's, so that we may now compute the functions more accurately at $t = -0.1$ and $+0.1$. We see that these are changed only slightly from their former values. A recomputation of ${}^{II}f_0$ and ${}^{I}f_0$ does not change their values, so that they are now final

and we may proceed to build up the table. It should be noted that, since the function is multiplied by h^2 throughout, the value of the single integral given by ((1,11)) is now $h(D)^{-1}\{f_l\}$, not $(hD)^{-1}\{f_l\}$. In the present example, this means that the velocity of x is given in units of t = 0.1. The completed example follows:

t	2nd Sum.	1st Sum.	Fn.	1st Diff.
-0.1	1.9917		-0.0199	
		+0.0100		- 1
0.0	2.0017		-0.0200	
		-0.0100		+ 1
0.1	1.9917		-0.0199	
		-0.0299		+ 3
0.2	1.9618		-0.0196	
		-0.0495		+ 5
0.3	1.9123		-0.0191	
		-0.0686		+ 7
0.4	1.8437		-0.0184	
		-0.0870		+ 8
0.5	1.7567		-0.0176	
		-0.1046		+11
0.6	1.6521		-0.0165	
		-0.1211		+12
0.7	1.5310		-0.0153	
		-0.1364		+14
0.8	1.3946		-0.0139	

t	2nd Sum.	1st Sum.	Fn.	1st Diff.
0.8	1.3946		-0.0139	
		-0.1503		+15
0.9	1.2443		-0.0124	
		-0.1627		+16
1.0	1.0816		-0.0108	
		-0.1735		+17
1.1	0.9081		-0.0091	
		-0.1826		+19
1.2	0.7255		-0.0072	
		-0.1898		+18
1.3	0.5357		-0.0054	
		-0.1952		+20
1.4	0.3405		-0.0034	
		-0.1986		+20
1.5	0.1419		-0.0014	
		-0.2000		+20
1.6	-0.0581		+0.0006	
		-0.1994		
1.7	-0.2575			

It may be observed that the solution of this differential equation with the given initial values is $x = 2\cos t$. Therefore if we inversely interpolate for the value of t at which x vanishes, the solution is known to be $t = \frac{1}{2}\pi$, and we have an independent method for the computation of π.

None of the effects of higher order differences can be observed from such a simple 4-place computation. We shall therefore repeat the example with an 8-place computation. Start with the approximate values for the function column which we have from the 4-place example and fill in zeros to complete the 8-place values. Using ((1,10)), compute ${}^{\text{II}}f_0 = 2.00166750$; also ${}^{\text{I}}f_0 = 0$. These will enable us to derive new values for x, and then the new values of the functions, starting at t = 0 and placed symmetrically on either side, are -0.02000000, -0.01990008, -0.01960133. Write these values in place of the previous approximate values, and form the new differences. Then ((1,10)) shows that ${}^{\text{II}}f_0$ requires no further correction and so we may proceed to extend the table forward.

The final value of each quantity in the function column must, strictly speaking, be obtained by successive approximations, but the first value will be final if x is extrapolated with sufficient accuracy. For this purpose we must use ((1,10)), but we may eliminate the quantities "on the line" which are not known at this stage in terms of the known quantities "up the diagonal", (or "down the diagonal" if we are working backwards). Thus, we substitute

$$f_l = f_{l-1} + \Delta^{\text{i}}_{l-3/2} + \Delta^{\text{ii}}_{l-2} + \Delta^{\text{iii}}_{l-5/2} + \Delta^{\text{iv}}_{l-3} + \dots, \text{ etc.}$$

Then

$$\begin{aligned} D^{-2}\{f_l\} = {}^{\text{II}}f_l &+ 0.083333\, f_{l\mp1} \pm 0.08333\,\Delta^{\text{i}}_{l\mp3/2} = {}^{\text{II}}f_l + 0.083333\, f_l \\ &+ 0.079167\Delta^{\text{ii}}_{l\mp2} \pm 0.075\ \Delta^{\text{iii}}_{l\mp5/2} \qquad - 0.004167\,\Delta^{\text{ii}}_{l\mp1} \mp 0.004167\,\Delta^{\text{iii}}_{l\mp3/2} \\ &+ 0.07135\ \Delta^{\text{iv}}_{l\mp3} \pm 0.0682\ \Delta^{\text{v}}_{l\mp7/2} \qquad - 0.003654\,\Delta^{\text{iv}}_{l\mp2} \mp 0.00314\ \Delta^{\text{v}}_{l\mp5/2} \qquad ((1,12)) \\ &+ 0.065\ \Delta^{\text{vi}}_{l\mp4} \pm 0.06\ \Delta^{\text{vii}}_{l\mp9/2} \qquad - 0.0027\ \Delta^{\text{vi}}_{l\mp3} \mp \dots\dots \end{aligned}$$

The formula on the right is to be used after the function has once been computed, either to check the accuracy of the previous extrapolation of the integral or to enable a closer second approximation to be recomputed. This topic and others of interest in this type of work are discussed by Bower in the Lick Observatory Bulletin 445.

The final integration table is shown on the next page. The quantity in parentheses behind each value of the function is the correction in units of the 9th decimal which is required to give the function accurately to one more place, in other words, it is the negative of the rounding-off error which has been committed in each individual computation. The lack of smoothness in the higher order difference columns is caused by the accumulation of these rounding-off errors. For example, the value of each quantity in the fifth difference column is roughly -0.01 of the value in the third difference column, but the value of $\Delta^{\text{v}}_{1.45} = +10$ includes the following combination of these errors: 1(-1) - 5(0) + 10(+5) - 10(-4) + 5(+2) - 1(+4) = +95 in units of the 9th decimal place. If this were applied, it would restore this quantity to its proper, smooth value. In practice, one can not

afford to spend too much time analyzing the dropped figures, but it is necessary to discern the distinction between their effect and the presence of a real error, which should be detected by the difference check. The practice of writing + and - signs or high and low dots after the computed quantities to indicate a large rounding-off error, say between 0.3 and 0.5 in units of the last place, will make it easier to decide when the lack of smoothness is due to the combination of successive errors of opposite sign, and when it is not.

t	2nd Sum.	1st Sum.	Fn.	1st Diff.	2nd Diff.	3rd Diff.	4th Diff.	5th Diff.
-0.2	1.96176742		-0.01960133 (-2)		+19585		-196	
		+0.02990008		- 29875		+ 298		- 1
-0.1	1.99166750		-0.01990008 (-3)		+19883		-197	
		+0.01000000		- 9992		+ 101		- 5
0.0	2.00166750		-0.02000000 (0)		+19984		-202	
		-0.01000000		+ 9992		- 101		+ 5
0.1	1.99166750		-0.01990008 (-3)		+19883		-197	
		-0.02990008		+ 29875		- 298		+ 1
0.2	1.96176742		-0.01960133 (-2)		+19585		-196	
		-0.04950141		+ 49460		- 494		+ 5
0.3	1.91226601		-0.01910673 (0)		+19091		-191	
		-0.06860814		+ 68551		- 685		+ 7
0.4	1.84365787		-0.01842122 (0)		+18406		-184	
		-0.08702936		+ 86957		- 869		+ 9
0.5	1.75662851		-0.01755165 (-1)		+17537		-175	
		-0.10458101		+104494		-1044		+10
0.6	1.65204750		-0.01650671 (-3)		+16493		-165	
		-0.12108772		+120987		-1209		+12
0.7	1.53095978		-0.01529684 (-4)		+15284		-153	
		-0.13638456		+136271		-1362		+15
0.8	1.39457522		-0.01393413 (-5)		+13922		-138	
		-0.15031869		+150193		-1500		+14
0.9	1.24425653		-0.01243220 (0)		+12422		-124	
		-0.16275089		+162615		-1624		+13
1.0	1.08150564		-0.01080605 (+3)		+10798		-111	
		-0.17355694		+173413		-1735		+25
1.1	0.90794870		-0.00907192 (-3)		+ 9063		- 86	
		-0.18262886		+182476		-1821		+11
1.2	0.72531984		-0.00724716 (+4)		+ 7242		- 75	
		-0.18987602		+189718		-1896		+20
1.3	0.53544382		-0.00534998 (+2)		+ 5346		- 55	
		-0.19522600		+195064		-1951		+26
1.4	0.34021782		-0.00339934 (-4)		+ 3395		- 29	
		-0.19862534		+198459		-1980		+10
1.5	0.14159248		-0.00141475 (+5)		+ 1415		- 19	
		-0.20004009		+199874		-1999		
1.6	-0.05844761		+0.00058399 (0)		- 584			
		-0.19945610		+199290				
1.7	-0.25790371		+0.00257689 (-1)					

A development similar to that given above will enable us to obtain formulas for the integrals and derivatives of a function at points midway between the tabular values of the argument. Define two new operators, similar to ((1,3)) and ((1,4)), such that

$$\Delta\{f_{l+1/2}\} = f_{l+1} - f_l = \Delta^{I}_{l+1/2}, \qquad \underline{\Delta}^2\{f_{l+1/2}\} = \tfrac{1}{2}(f_{l+2} - f_{l+1} - f_l + f_{l-1}) = \Delta^{II}_{l+1/2}.$$

Then $$\Delta = (e^{hD/2} - e^{-hD/2}) \qquad \underline{\Delta}^2 = \tfrac{1}{2}(e^{hD/2} + e^{-hD/2})(e^{hD/2} - e^{-hD/2})^2$$

It will be observed that the effect of writing the nth order differences as $(e^{hD/2} - e^{-hD/2})^n$ is to express them in terms of the functions, as can be seen by expanding the binomial and considering the symbolic expression of Taylor's series as used on page 6. The effect of $\frac{1}{2}(e^{hD/2} + e^{-hD/2})$ is for the first term to lower all the functions a half line in the table and the second term is to raise them a half line; then the mean is taken. This is equivalent to taking the mean of the differences on the half line below and the half line above, as described in the paragraph following ((1,1)). This time it is the even order difference columns in which we are obliged to form the mean, and it is therefore the even powers of (hD) which need the factor $\sqrt{1 + \Delta^2/4}$ in the denominator.

The student will supply all the intervening steps; the development is exactly analogous to that of the preceding case. We obtain

$$(hD)^{2k+1} = \left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k+1}, \quad (hD)^{2k} = \frac{1}{\sqrt{1 + \Delta^2/4}}\left(\int\frac{d\Delta}{\sqrt{1 + \Delta^2/4}}\right)^{2k}$$

and in deriving the corresponding formulas for the coefficients we have only to make the proper changes in the exponents. Then

$$(hD)^{2k+1} = (2k + 1)\int(hD)^{2k}\, d\Delta, \qquad A^{(2k)}_{j+2} = \frac{2k}{j + 2}A^{(2k-1)}_{j+1} - \frac{j + 1}{4(j + 2)}A^{(2k)}_{j} .$$

The initial coefficients are obtained from

$$(hD) = \int \frac{d\Delta}{\sqrt{1+\Delta^2/4}} = \Delta + \sum_{j=1}^{\infty} (-1)^j \frac{1\ 3\ 5\ .\ .\ (2j-1)}{j!\,(2j+1)\,2^{2j}} \Delta^{2j+1} = \Delta - \frac{1}{24}\Delta^3 + \frac{3}{640}\Delta^5 - \dots$$

It will be a valuable exercise for the student to verify the following results:

$$\begin{aligned}
hD &= \Delta - \frac{1}{24}\Delta^3 + \frac{3}{640}\Delta^5 - \frac{5}{7168}\Delta^7 + \frac{35}{294912}\Delta^9 - \dots \\
(hD)^2 &= \Delta^2 - \frac{5}{24}\Delta^4 + \frac{259}{5760}\Delta^6 - \frac{3229}{322560}\Delta^8 + \dots \\
(hD)^3 &= \Delta^3 - \frac{1}{8}\Delta^5 + \frac{37}{1920}\Delta^7 - \frac{3229}{967680}\Delta^9 + \dots \\
(hD)^4 &= \Delta^4 - \frac{7}{24}\Delta^6 + \frac{47}{640}\Delta^8 - \dots \\
(hD)^5 &= \Delta^5 - \frac{5}{24}\Delta^7 + \frac{47}{1152}\Delta^9 - \dots \\
(hD)^6 &= \Delta^6 - \frac{3}{8}\Delta^8 + \dots
\end{aligned} \qquad ((1,13))$$

The series for $(hD)^{-1}$ is easily obtained by taking the reciprocal of the (hD) series, and then

$$(hD)^{-2} = -\frac{d}{d\Delta}(hD)^{-1}.$$

$$(hD)^{-1} = \Delta^{-1} + \frac{1}{24}\Delta - \frac{17}{5760}\Delta^3 + \frac{367}{967680}\Delta^5 - \frac{27859}{464486400}\Delta^7 + \dots \qquad ((1,14))$$

$$(hD)^{-2} = \Delta^{-2} - \frac{1}{24} + \frac{17}{1920}\Delta^2 - \frac{367}{193536}\Delta^4 + \frac{27859}{66355200}\Delta^6 - \dots \qquad ((1,15))$$

These formulas, it must be recalled, all refer to quantities "on the half line" since the operation is upon $f_{l+1/2}$.

We are now prepared to attack the more general problem of interpolation to obtain the value of the function corresponding to any value of the argument within the interval. For this purpose we shall again use a Taylor's series expansion. From the point of view of the differential calculus this is, in effect, a set of Taylor's series within a Taylor's series. Write

$$\begin{aligned}
f(t_l + nh) &= e^{nhD}\{f_l\} = (1 + n(hD) + \frac{n^2}{2!}(hD)^2 + \frac{n^3}{3!}(hD)^3 + \dots)\{f_l\} \\
&= f_l + n(\Delta_l^{\mathrm{i}} - \frac{1}{6}\Delta_l^{\mathrm{iii}} + \frac{1}{30}\Delta_l^{\mathrm{v}} - \dots) \\
&\quad + \frac{n^2}{2!}(\Delta_l^{\mathrm{ii}} - \frac{1}{12}\Delta_l^{\mathrm{iv}} + \frac{1}{90}\Delta_l^{\mathrm{vi}} - \dots) \\
&\quad + \frac{n^3}{3!}(\Delta_l^{\mathrm{iii}} - \frac{1}{4}\Delta_l^{\mathrm{v}} + \dots) \\
&\quad + \frac{n^4}{4!}(\Delta_l^{\mathrm{iv}} - \frac{1}{6}\Delta_l^{\mathrm{vi}} + \dots) \\
&\quad + \dots \\
&= f_l + n\Delta_l^{\mathrm{i}} + \frac{n^2}{2!}\Delta_l^{\mathrm{ii}} + \frac{n(n^2-1^2)}{3!}\Delta_l^{\mathrm{iii}} + \frac{n^2(n^2-1^2)}{4!}\Delta_l^{\mathrm{iv}} + \frac{n(n^2-1^2)(n^2-2^2)}{5!}\Delta_l^{\mathrm{v}} + \dots
\end{aligned} \qquad ((1,16))$$

This is known as STIRLING's Formula. It depends upon the horizontal differences of all orders "on the line" and opposite one of the tabular values of the argument. From this we may derive another formula which depends upon the even order differences of two successive lines. Let $m = 1 - n$, and substitute $\Delta_l^{(2k+1)} = \Delta_{l+1}^{(2k)} - \Delta_l^{(2k)} - \frac{1}{2}\Delta_l^{(2k+2)}$. Then

$$\begin{aligned}
f(t_l + nh) &= m f_l + \frac{m(m^2-1^2)}{3!}\Delta_l^{\mathrm{ii}} + \frac{m(m^2-1^2)(m^2-2^2)}{5!}\Delta_l^{\mathrm{iv}} + \dots \\
&\quad + n f_{l+1} + \frac{n(n^2-1^2)}{3!}\Delta_{l+1}^{\mathrm{ii}} + \frac{n(n^2-1^2)(n^2-2^2)}{5!}\Delta_{l+1}^{\mathrm{iv}} + \dots
\end{aligned} \qquad ((1,17))$$

This is known as EVERETT's Formula. It has the advantage of requiring the printing of only half

as many columns of differences and the tabulation of only half as many coefficients as other formulas of the same accuracy. In the special case of $n = \frac{1}{2}$, we have

$$f(t_i + \tfrac{1}{2}h) = f_{i+1/2} - \frac{1}{8}\Delta^{ii}_{i+1/2} + \frac{3}{128}\Delta^{iv}_{i+1/2} - \frac{5}{1024}\Delta^{vi}_{i+1/2} + \ldots$$

In a similar manner we may develop a formula in terms of the differences "on the half line". (For convenience, the subscript $i+1/2$ has been omitted from all the quantities which are to be taken from the numerical table.)

$$\begin{aligned} f(t_i+\tfrac{1}{2}h+(n-\tfrac{1}{2})h) &= e^{(n-1/2)hD}\{f_{i+1/2}\} = \left(1 + (n-\tfrac{1}{2})(hD) + \frac{(n-\frac{1}{2})^2}{2!}(hD)^2 + \ldots\right)\{f_{i+1/2}\} \\ &= f \quad - \frac{1}{8}\Delta^{ii} \quad + \frac{3}{128}\Delta^{iv} \quad - \ldots \\ &\quad + (n-\tfrac{1}{2})(\Delta^{i} \quad - \frac{1}{24}\Delta^{iii} \quad + \frac{3}{640}\Delta^{v} \quad - \ldots) \\ &\quad + \frac{(n-\frac{1}{2})^2}{2!}(\Delta^{ii} \quad - \frac{5}{24}\Delta^{iv} \quad + \ldots) \\ &\quad + \frac{(n-\frac{1}{2})^3}{3!}(\Delta^{iii} \quad - \frac{1}{8}\Delta^{v} \quad + \ldots) \\ &\quad + \ldots \qquad\qquad (1,18) \\ &= f + (n-\tfrac{1}{2})\Delta^{i} + \frac{n(n-1)}{2!}\Delta^{ii} + (n-\tfrac{1}{2})\frac{n(n-1)}{3!}\Delta^{iii} + \frac{n(n^2-1)(n-2)}{4!}\Delta^{iv} + \ldots \end{aligned}$$

This is known as BESSEL's Formula.

The same principle may be applied for the derivation of direct interpolation formulas for integrals or derivatives as well as for the function. Write

$$(hD)^k\{f_{i+nh}\} = e^{nhD}\{(hD)^k\{f_i\}\} = \left((hD)^k + n(hD)^{k+1} + \frac{n^2}{2!}(hD)^{k+2} + \frac{n^3}{3!}(hD)^{k+3} + \ldots\right)\{f_i\}$$

and substitute as before. We shall give one example in detail, for the case $k = -2$:

$$\begin{aligned} \iint^{t_i+nh} f(t)\,dt^2 &= {}^{ii}f_i \quad + \frac{1}{12}f_i \quad - \frac{1}{240}\Delta^{ii}_i \quad + \ldots \\ &\quad + n({}^{i}f_i \quad - \frac{1}{12}\Delta^{i}_i \quad + \frac{11}{720}\Delta^{iii}_i \quad - \ldots) \\ &\quad + \frac{n^2}{2!}(f_i) \\ &\quad + \frac{n^3}{3!}(\Delta^{i}_i \quad - \frac{1}{6}\Delta^{iii}_i \quad + \ldots) \\ &\quad + \ldots \qquad\qquad (1,19) \\ &= {}^{ii}f_i + n\,{}^{i}f_i + \left(\frac{n^2}{2} + \frac{1}{12}\right)f_i + \left(\frac{n^3}{6} - \frac{n}{12}\right)\Delta_i + \left(\frac{n^4}{24} - \frac{1}{240}\right)\Delta^{ii}_i + \left(\frac{n^5}{120} - \frac{n^3}{36} + \frac{11n}{720}\right)\Delta^{iii}_i + \ldots \end{aligned}$$

This formula bears the same relationship to the interpolation of a table of double integration that Stirling's formula does to the interpolation of a function from its tabulated values.

By means of the same substitution as before, we may derive a formula which is similar to Everett's formula.

$$\begin{aligned} \iint^{t_i+nh} f(t)\,dt^2 = \; & m\,{}^{ii}f_i + \frac{m}{12}(2m^2-1)f_i + \frac{m}{720}(6m^4 - 20m^2 + 11)\Delta^{ii}_i + \ldots \\ & + n\,{}^{ii}f_{i+1} + \frac{n}{12}(2n^2-1)f_{i+1} + \frac{n}{720}(6n^4 - 20n^2 + 11)\Delta^{ii}_{i+1} + \ldots \end{aligned} \qquad (1,20)$$

In this equation, the variable is n, so that if we differentiate with respect to n, we have $dt = h\,dn$, and thus we obtain

$$\begin{aligned}\int^{t_i+nh} f(t)\,dt = {}^{\mathrm{I}}f_{i+1/2} &- \frac{1}{12}(6m^2-1)f_i - \frac{1}{720}(30m^4-60m^2+11)\Delta_i^{\mathrm{II}} - \ldots \\ &+ \frac{1}{12}(6\,n^2-1)f_{i+1} + \frac{1}{720}(30\,n^4-60\,n^2+11)\Delta_{i+1}^{\mathrm{II}} + \ldots \end{aligned} \qquad ((1{,}21))$$

It must be understood that in using this last formula the function column must contain hf, and not h^2f, as it would for a double integration table; if the latter case obtained, the result would have to be divided by h. The coefficients for these two formulas are tabulated in the appendix.

As an exercise, the student may derive the following formulas, all of which depend upon the quantities "on the half line" in the table. (For convenience, the subscript $i+1/2$ has been omitted.)

$$\begin{aligned}\iint^{t_i+nh} f(t)\,dt &= {}^{\mathrm{II}}f + (n-\tfrac{1}{2})\,{}^{\mathrm{I}}f + \left(\frac{(n-\frac{1}{2})^2}{2} - \frac{1}{24}\right)f + \left(\frac{(n-\frac{1}{2})^3}{6} + \frac{(n-\frac{1}{2})}{24}\right)\Delta^{\mathrm{I}} + \\ &\left(\frac{(n-\frac{1}{2})^4}{24} - \frac{(n-\frac{1}{2})^2}{16} + \frac{17}{1920}\right)\Delta^{\mathrm{II}} + \left(\frac{(n-\frac{1}{2})^5}{120} - \frac{(n-\frac{1}{2})^3}{144} - \frac{17(n-\frac{1}{2})}{5760}\right)\Delta^{\mathrm{III}} + \ldots\end{aligned}$$

$$\int^{t_i+nh} f(t)\,dt = {}^{\mathrm{I}}f + (n-\tfrac{1}{2})f + \left(\frac{(n-\frac{1}{2})^2}{2} + \frac{1}{24}\right)\Delta^{\mathrm{I}} + \left(\frac{(n-\frac{1}{2})^3}{6} - \frac{(n-\frac{1}{2})}{8}\right)\Delta^{\mathrm{II}} + \ldots$$

$$hD\{f_{i+nh}\} = \Delta^{\mathrm{I}} + (n-\tfrac{1}{2})\Delta^{\mathrm{II}} + \left(\frac{(n-\frac{1}{2})^2}{2} - \frac{1}{24}\right)\Delta^{\mathrm{III}} + \left(\frac{(n-\frac{1}{2})^3}{6} - \frac{5(n-\frac{1}{2})}{24}\right)\Delta^{\mathrm{IV}} + \ldots$$

Before concluding the subject of interpolation, we shall describe the principle of the "throw-back". It will be observed that in Bessel's formula the coefficient $B^{\mathrm{IV}} = \frac{(n+1)(n-2)}{12}B^{\mathrm{II}}$ and in the interval from 0 to 1, the factor multiplying B^{II} is nearly constant. Adopt the value -0.184, and then write $M_i = \Delta_i^{\mathrm{II}} - 0.184\,\Delta_i^{\mathrm{IV}}$. This "modified second difference" may be used instead of Δ_i^{II} in either Bessel's or Everett's formulas with the result that the fourth difference effect is automatically included with the second difference terms. It is a valuable exercise for the student to derive the value -0.184 independently before reading further, and to test the error of this approximation at various points throughout the interval from 0 to 1. The error is as large as one unit in the last place when the fourth difference is as large as 2300 units in the last place. Other "throwbacks" may be derived for other formulas and other orders of differences.

If we wish to derive the proper constant value to be used in our "throwback" approximation, the error we commit may be written as $\Delta_i^{\mathrm{IV}}(KB^{\mathrm{II}} - B^{\mathrm{IV}})$. To keep this error down to a minimum, irrespective of the values which n may take within the interval from 0 to 1, we may impose the condition that the sum of the squares of the errors shall be a minimum. Since K is the only variable at our disposal, the sum of the squares will be a minimum when its derivative with respect to K is zero, i.e. when $K\sum(B^{\mathrm{II}})^2 = \sum B^{\mathrm{II}}B^{\mathrm{IV}}$. If we evaluate this equation for K at intervals of 0.1 in n, and take the summations, we obtain K = 0.18453. But strictly, we should evaluate the equation for K at infinitesimally small intervals of n, and therefore in the limit we must replace the summations by integrals and we have $K\int_0^1(B^{\mathrm{II}})^2dn = \int_0^1 B^{\mathrm{II}}B^{\mathrm{IV}}dn$, or K = -31/168 = 0.18452. In this case the integrals may be evaluated analytically by substituting for the B's in terms of n, or they may be evaluated numerically by means of ((1,11)). The latter course will exhibit the corrections which the higher order terms produce or, what amounts to the same thing, it will indicate the error committed in replacing an integral by a summation, a practice frequently employed in applications. If K is determined from any other reasonable assumptions, the resulting value is very nearly the same. For example, if we impose the condition that the maximum positive and negative errors shall be of equal absolute magnitude, then K comes out slightly less than -0.184.

Subtabulation is closely related to the topics we have been considering. We shall outline briefly the lines along which the reader may develop for himself the very efficient method which is due to Comrie. We shall assume the case in which the 4th differences of the original table are less than 1000 units in the last place, and the subdivision to 10ths is required. Apply the "throw-back" from the 4th differences into the 2nd, but always round off to the closest even number in the last digit. Write the expression for each of the interpolates to 10ths, using Everett's formula with modified second differences, and inserting the exact numerical values of the Everett coeffi-

cients, but keeping the differences literal. Now difference these literal expressions for the values of the functions in the subdivided table, until the 4th differences are reached. Since the Everett coefficients are cubic expressions, they have no 4th differences, and therefore all the 4th differences of the subdivided table will be zero except three bridging values which we reach when we cross from one interval to the next. Also our literal differences give us formulas for the leading differences in each column of the subdivided table. If we use the notation F and Δ for the original table, and f and δ for the subdivided table, and if we write

$$M_i = \Delta_i^{ii} - 0.184\Delta_i^{iv} = \Delta_i^{ii} - D_i^{iv}, \quad \text{and } D_i^{vi} = \text{2nd diff. of } D_i^{iv},$$

then we have

$$\begin{aligned} f_{-1} &= +0.1F_{-1} + 0.9F_0 - 0.0165M_{-1} - 0.0285M_0 \\ f_0 &= \phantom{+0.1F_{-1}} + 1.0F_0 \\ f_1 &= +0.9F_0 + 0.1F_1 - 0.0285M_0 - 0.0165M_1 \\ f_2 &= +0.8F_0 + 0.2F_1 - 0.048\ M_0 - 0.032\ M_1, \text{ etc.} \end{aligned} \qquad (1,22)$$

$$\begin{aligned} \delta_{1/2}^{i} &= +0.1\Delta_{1/2}^{i} - 0.0285M_0 - 0.0165M_1 & \delta_{i-1}^{iv} &= -0.0165(\Delta_i^{iv} - D_i^{vi}) + 0.1D_i^{iv} \\ \delta_1^{ii} &= +0.009\ M_0 + 0.001\ M_1 & \delta_i^{iv} &= +0.034\ (\Delta_i^{iv} - D_i^{vi}) - 0.2D_i^{iv} \\ \delta_{3/2}^{iii} &= -0.001\ M_0 + 0.001\ M_1 & \delta_{i+1}^{iv} &= -0.0165(\Delta_i^{iv} - D_i^{vi}) + 0.1D_i^{iv} \end{aligned}$$

It will be observed that by carrying three extra decimal places beyond the end figure of the original table, all the values in the subdivided table can be retained exactly. The essence of the process then consists in computing all the non-zero bridging values in the 4th differences and using these to build up the 3rd differences, then the 2nd, and the 1st, and finally the function. All this is done while retaining the three extra decimal places. There is a rigid check on the work, for every tenth value in the subdivided table must reproduce exactly the corresponding value from the original table.

We have now developed numerical methods for obtaining the value of a function, its integrals, and its derivatives corresponding to any value of the argument. Any function which is continuous, no matter how complicated its analytical expression, is amenable to this treatment. It is much more profitable for the student to be familiar with these completely general methods than to be limited to the use of such approximate methods as Simpson's rule or the other rules which are usually taught in the regular courses in calculus. It is illuminating to examine the error of Simpson's rule in the light of the above formulas. If we integrate between the limits (i-h) to (i+h), the error is found to be

$$({}^{I}f_{i+1} - {}^{I}f_{i-1}) - \frac{1}{12}(\Delta_{i+1}^{i} - \Delta_{i-1}^{i}) + \frac{11}{720}(\Delta_{i+1}^{iii} - \Delta_{i-1}^{iii}) - \ldots - 3(f_{i+1} + 4f_i + f_{i-1}) = -\frac{1}{90}\Delta_i^{iv} + \ldots,$$

and if the integration is over a large number of intervals, say from 0 to 2j, the required correction to the value given by Simpson's rule is: $-\frac{1}{90}\sum_j \Delta_{2j-1}^{iv} +$ terms of higher even order differences. This can be verified by substituting for all odd order quantities in terms of those of even order and grouping properly. This result may be tested numerically by integrating t^5 from 0 to 1 by means of Simpson's rule, using h = 0.1. The value obtained is too large by 0.001/30, and the sum of the five alternate fourth differences (allowing for the factor h by which the function should be multiplied for a single integration table) is 0.003. Since there are no sixth or higher order differences in this example, the agreement is exact.

One further matter requires consideration, and that is the value of the interval h which is to be adopted. This always depends upon the particular problem and can not be covered by any general statement except that it should be small enough to cause the highest order difference to be reduced to about ten units in the last place. It is therefore a matter of practical importance to consider the problem of halving or doubling the interval. We shall consider in detail the case of a table of double integration.

Let the values in the function column of the table with interval h be denoted by F_i, and those in the table with interval $\frac{1}{2}$h by f_i. As in the previous developments, we shall define two symbolic operators, Δ^2 and δ^2, in such a way that

$$\Delta^2 = (e^{hD} - 2 + e^{-hD}) = (e^{hD/2} - e^{-hD/2})^2, \qquad \delta^2 = (e^{hD/2} - 2 + e^{-hD/2}) = (e^{hD/4} - e^{-hD/4})^2.$$

Then $\Delta^2 = \delta^2(4 + \delta^2)$, $\quad(\delta^2 + 2)^2 = \Delta^2 + 4$, $\quad\delta^2 = \sqrt{4 + \Delta^2} - 2$,

$$\delta^{-2} = \frac{\sqrt{4+\Delta^2}+2}{\Delta^2} = \frac{2}{\Delta^2}\left(1+\sqrt{1+\Delta^2/4}\right) = \frac{4}{\Delta^2}\left(1+\frac{\Delta^2}{2^4}-\frac{\Delta^4}{2^8}+\frac{\Delta^6}{2^{11}}-\frac{5\Delta^8}{2^{16}}+\ldots\right).$$

Now ${}^{II}F_1 = \Delta^{-2}\{F_1\}$, ${}^{II}f_1 = \delta^{-2}\{f_1\}$, and $F_1 = 4f_1$. Therefore

$$\delta^{-2}\{f_1\} = \left(\Delta_1^{-2}+\frac{1}{2^4}-\frac{\Delta^2}{2^8}+\frac{\Delta^4}{2^{11}}-\frac{5\Delta^6}{2^{16}}+\ldots\right)\{4f_1\}$$

or

$$ {}^{II}f_1 = {}^{II}F + \frac{F_1}{2^4} - \frac{\Delta_1^{II}}{2^8} + \frac{\Delta_1^{IV}}{2^{11}} - \frac{5\Delta_1^{VI}}{2^{16}} + \ldots \qquad ((1,23))$$

The author is indebted to J.C.P.Miller for this demonstration by means of symbolic operators.

Let us consider two more numerical exercises based on our 8-place integration table on page 11. First, let us use the coefficients tabulated in the appendix to make an accurate inverse interpolation for $\frac{1}{2}\pi$. Write ((1,20)) in the form

$$-{}^{I}f_{1+1/2}n = {}^{II}f_1 + \frac{m}{12}(2m^2-1)f_1 + \ldots$$
$$+ \frac{n}{12}(2n^2-1)f_{1+1} + \ldots$$

and solve for n by iteration. Begin with $n = -{}^{II}f_1/{}^{I}f_{1+1/2} = +0.708$, and take out the coefficients with this argument. Since this is only a rough first approximation, we choose a value which requires no interpolation. Recompute n and repeat the iterative process until the solution converges to the final value. The individual quantities for the last approximation are:

$$+0.20004009\,n = +0.14159248 - 0.0201853(-141475) + 0.003787(+1415)$$
$$+ 0.0001432(+58399) + 0.002442(-584)$$

and $n = 0.7079639$. This gives $\frac{1}{2}\pi = 1.57079639$, which is 6 units too large in the last place. The discordance is due to the accumulation of the rounding off errors. This can be overcome only by carrying more decimal places or increasing the interval. The reader will find an excellent discussion of this subject by Brouwer: On the Accumulation of Errors in Numerical Integration in the Astronomical Journal, vol. 46, p. 149.

Second, let us halve the interval of this table, beginning at $t_1 = 1.5$. By means of ((1,22)), compute f_j for $t_j = 1.4, 1.5, 1.6$; and by means of ((1,15)), compute x and f_j at $t_j = 1.45, 1.55$. Then

$${}^{I}f_{1+1/2} = \tfrac{1}{2}({}^{II}f_{1+2} - {}^{II}f_1 - f_{1+1}), \qquad {}^{I}f_{1-1/2} = \tfrac{1}{2}({}^{II}f_1 - {}^{II}f_{1-2} + f_{1-1})$$

and, as a check, their difference should be equal to f_1.

It must be recognized that this check cannot always be exact, due to the accumulation of the rounding off errors in the end figures. An independent check is obtained by evaluating the first and second integrals from both the table with the divided and the undivided interval at points which they have in common. Based on the extent to which these do not agree, the end figures of the values in the 1st and 2nd Sum. columns of the new table might well be adjusted one or two units. If too large an adjustment is indicated, this is likely to be due to some error in the work, or the higher order differences are already too large and the subdivision should have been made sooner.

The subdivided table then appears as follows:

t	2nd Sum.	1st Sum.	Fn.	1st Diff.	2nd Diff.	3rd Diff.
1.40	0.34000523		-84984			
		-0.09894936		+24733		
1.45	0.24105587		-60251		+149	
		-0.09955187		+24882		-59
1.50	0.14150400		-35369		+ 90	
		-0.09990556		+24972		-65
1.55	0.04159844		-10397		+ 25	
		-0.10000953		+24997		
1.60	-0.05841109		+14600			

When the initial conditions are given for some particular value of the argument t_0, there is the option of choosing to build the table so that $t_0 = t_i$ or $t_0 = t_{i+1/2}$. It will be observed that in the series ((1,10)), ((1,11)), ((1,14)), ((1,15)), the coefficients converge most rapidly for ((1,14)) and least rapidly for ((1,11)), but ((1,10)) converges more rapidly than ((1,15)). It is therefore in all cases preferable to use $t_0 = t_{i+1/2}$ unless some other factor overrules this consideration. If t_0 is necessarily at some fractional point within the interval, the starting values in the 1st and 2nd Sum. columns can be determined by using two of the appropriate interpolation formulas, but with ^{II}f or ^{I}f standing in the equations as unknowns and the numerical values of the initial conditions substituted on the lefthand side. Usually ((1,21)) and then ((1,20)) will be found most convenient for this purpose.

This concludes the demonstrations that will be presented on this subject. They have been given with increasing brevity in order that the student may gain increasing experience with these methods of solution through his own resources and confidence to attack successfully such other related problems as may arise. The subject might be expanded still further, particularly in the direction of analyzing approximate methods of integration or of evaluating certain types of definite integrals. The contents of this chapter will be needed for the interpolation of the solar coordinates and some of the abbreviated tables in the appendix, and also for the integrations required in the special perturbations.

CHAPTER 2

PROBLEMS IN SPHERICAL ASTRONOMY

Πρόεσθ' ἐλπίδα πάντες ἀφικνούμενοι.

Before entering upon the discussion of the main problem, we shall consider briefly several problems in spherical astronomy which will be required in our later work. To those who are not familiar with this subject it must be emphasized at the start that, even though we are dealing with three dimensional space, an astronomical observation of a celestial object is limited to the determination of two angular coordinates upon the sky, but nothing can be determined by observation about the distance of the object along the line of sight. As viewed from the Earth, the situation is then the same as if we were dealing with the motion of a point that is constrained to move upon the surface of a unit sphere which is centered at the observer, and which is known in Astronomy as the celestial sphere.

The Earth is revolving about the Sun in a period of a year, and it is also rotating on its axis in a period of about four minutes less than a day. If the reader will imagine himself situated as an observer at the center of the Earth (where he will not be affected by its rotation) then in the course of a year the Sun will be seen to trace out a path among the stars in the sky which is a great circle known as the ecliptic. If the observer will also project out onto the sky, from his vantage point at the center of the Earth, the equator of the Earth, he will trace out another great circle known as the celestial equator. These two great circles intersect with a dihedral angle of about $23\frac{1}{2}°$, known as the obliquity. For the purposes of the subject which we shall treat in this volume, the stars may be considered to be so infinitely far away that they become practically fixed points of reference on the celestial sphere. The distances to the Sun and the objects which we shall study that are revolving around it are, however, large in comparison to the radius of the Earth. The small difference between the directions in which an object is seen from the center of the Earth and from some point on its surface is known as parallax.

The reader may now return from his ignominious position at the center of the Earth to the more familiar region on its surface. Due to the eastward rotation of the Earth, the objects on the sky appear to rotate westward about the poles of the celestial equator, or the north and south celestial poles. The most natural system of spherical coordinates in which to measure positions of objects upon the sky is that provided by this rotation. The fundamental plane is the celestial equator, and the angle in this coordinate between any two points may be determined simply by noting the amount of time elapsed between their respective transits over the observer's meridian. This system is known as the equatorial coordinate system.

Another system with which we shall have to deal is known as the ecliptic coordinate system, as it adopts the ecliptic, or the plane of the Earth's revolution around the Sun, as the fundamental plane. The fundamental planes of these two systems intersect at two diametrically opposite points and the intersection at which the Sun crosses the celestial equator from south to north is called the vernal equinox (♈). This is taken as the origin of coordinates in both systems.

The equatorial coordinates of any point S upon the sky may be defined by passing a great circle through S and the celestial poles, intersecting the celestial equator at F. Then the angle along the celestial equator from the vernal equinox to F is called the right ascension α, it is usually expressed in hours, minutes, and seconds of time. The angle FS, perpendicular to the celestial equator, is called the declination δ. It is usually expressed in degrees, minutes, and seconds of arc, positive to the north or negative to the south. The corresponding coordinates in the ecliptic system are known as celestial longitude and celestial latitude.

Unfortunately, the poles and the fundamental planes of both of these coordinate systems are

in continuous motion and the coordinates of the same point on the sky will be found to be different when measured at different times. The ecliptic is being moved very slightly due to the attractions of the other planets upon the Earth. The attractions of the Sun and the Moon upon the material in the region of the Earth's equator which is in excess of a true sphere cause the Earth's axis to precess or "wobble" slowly in space in a period of about 26,000 years, similar to the action of a spinning top. The total effect is divided into two parts: the short period, irregular part is called nutation, and the remaining part, which is nearly uniform, is called precession. The fictitious equator and equinox which partake of the precessional motion only are known as the mean equator and equinox of date. Since the reference system is in motion, the positions of an object which are measured at different times are not strictly comparable until they have been corrected for this motion during the intervening time.

The reduction from the observed or apparent position, which is naturally referred to the true equator and equinox at the time of observation, to the position referred to the mean equator and equinox at the beginning of the year is called the mean place reduction. It is now customary for the observer to apply this reduction before publishing his observations. Then, if observations from more than one year are to be used together, the computer must apply the appropriate reduction for precession in order to bring them all to the same mean equator and equinox. It is now customary, for the sake of uniformity and convenience, to use the mean equator and equinox of 1950.0. The British Nautical Almanac Office has also published two volumes entitled Planetary Coordinates, London, 1933 and 1939, which give their data referred to 1950.0.

The rates of change of the right ascension and declination due to precession are given by the following formulas:

$$\frac{d\alpha}{dt} = m + n\sin\alpha\tan\delta, \qquad \frac{d\delta}{dt} = n\cos\alpha, \qquad ((2,1))$$

where the values $m = +\,3\overset{s}{.}07327 + 0\overset{s}{.}0000186(t - 1950)$ and $n = +\,20\overset{''}{.}0426 - 0\overset{''}{.}000085(t - 1950)$ or $n = +\,1\overset{s}{.}33617 - 0\overset{s}{.}0000057(t - 1950)$

will give the rates per year. In principle, we have the problem of solving for two quantities (α, δ) by means of single numerical integrations, where the integrands depend upon the integrals, but in practice the integrated quantities usually change so uniformly that it is sufficient to compute the total change in each coordinate simply by multiplying the number of years in the interval by the rate at the middle of the interval. Since the coordinates at the middle of the interval are not known at the start, it is necessary to use the values for the beginning of the interval in order to get a start, and then proceed by successive approximations.

In the region of the poles, over very long intervals of time, or simply as a check, the integrations may be performed by Simpson's rule. In this case the value of the integrand at the middle of the interval is replaced by a weighted mean of the values at the beginning, middle, and end of the interval, where the weights are +1/6, +4/6, and +1/6, respectively. It is still necessary to proceed by successive approximations. Two types of expansions are also in common use:

$$\alpha = \alpha_0 + T\frac{d\alpha_0}{dt} + \frac{T^2}{2!}\frac{d^2\alpha_0}{dt^2} + \frac{T^3}{3!}\frac{d^3\alpha_0}{dt^3} + \ldots = \alpha_0 + A_0 + A_1\tan\delta_0 + A_2\tan^2\delta_0 + \ldots \qquad ((2,2))$$

and similarly for δ. The former is used in star catalogues and the coefficients are tabulated in Schorr's Prazessions-Tafeln, Bergedorf, 1927. Tables of the coefficients for the second formula have been given by Ristenpart, Publications of the Observatory of Santiago; those for the current year are published in the British Nautical Almanac, and others are in the back of the volumes of Planetary Coordinates.

The rectangular coordinates of a point referred to one coordinate system are some linear combination of those referred to any other system having the same origin. The numerical computation of such linear combinations is reduced to a convenient routine by the use of "Cracovians", a form of matrix multiplication in which the rules for multiplication provide that the elements be multiplied in pairs column by column instead of column by row. Thus if we write the equations

$$\begin{aligned} x &= X_x x_0 + Y_x y_0 + Z_x z_0 \\ y &= X_y x_0 + Y_y y_0 + Z_y z_0 \\ z &= X_z x_0 + Y_z y_0 + Z_z z_0 \end{aligned} \quad \text{as} \quad \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \end{Bmatrix} \begin{Bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{Bmatrix}. \qquad ((2,3))$$

it is evident what the rule for the multiplication is. In general, if the product of two Cracovians is represented as

$$\begin{Bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{Bmatrix} \begin{Bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \\ b_{13} & b_{23} \end{Bmatrix} = \begin{Bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \end{Bmatrix} \quad \text{then} \quad c_{ij} = \sum_k a_{ik} b_{jk} \qquad ((2,4))$$

These Cracovians may be used in theoretical developments, in fact they are applicable in any case where matrices may be employed, but for the purposes of this volume they will be restricted to the role of a convenient computing form. The numerical coefficients for the linear combinations which express the effects of precession from one epoch to another may be found in the last two references above, the Lick Observatory Bulletin 445, or the Bauschinger-Stracke Tafeln zur Theoretischen Astronomie, Leipzig, 1934. For reduction from (or to) 1950.0, the values are given by the following formulas:

$$\begin{aligned} X_x &= +\,1.0000\,0000 \;-\; 0.0002\,9696\,T^2 - 0.0000\,0014\,T^3 \\ Y_x = -X_y &= -\,0.0223\,4941\,T - 0.0000\,0676\,T^2 + 0.0000\,0221\,T^3 \\ Z_x = -X_z &= -\,0.0097\,1691\,T + 0.0000\,0206\,T^2 + 0.0000\,0098\,T^3 \\ Y_y &= +\,1.0000\,0000 \;-\; 0.0002\,4975\,T^2 - 0.0000\,0015\,T^3 \\ Y_z = +Z_y &= -\,0.0001\,0858\,T^2 \\ Z_z &= +\,1.0000\,0000 \;-\; 0.0000\,4721\,T^2 + 0.0000\,0002\,T^3 \end{aligned} \qquad ((2,5))$$

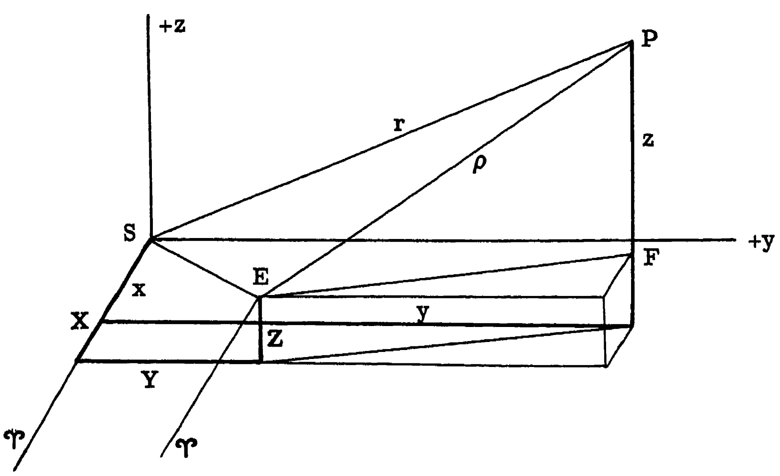
where T is measured in tropical centuries. If x, y, and z are given and x_0, y_0, and z_0 are to be found, the Cracovian must be reflected through its principal diagonal, i.e. interchange X_y and Y_x, and X_z and Z_x, as well as x_0 and x, etc.

A more detailed description of Cracovians and some other applications will be found in the Circulaire de l'Observatoire de Cracovie, No. 17, and in the Astronomical Journal, vol. 46, p. 132, and vol. 48, p. 105.

All our observations of celestial objects are made from the Earth, although the Sun is the predominant mass in the solar system and the most natural origin of coordinates. To translate the origin from the Earth to the Sun, let

$$\begin{aligned} x + X &= \rho\cos\delta\,\cos\alpha = \rho\, l \\ y + Y &= \rho\cos\delta\,\sin\alpha = \rho m \\ z + Z &= \rho\sin\delta \qquad\quad\; = \rho n \end{aligned} \qquad ((2,6))$$

where x, y, z are the heliocentric, equatorial, rectangular coordinates of the object; X, Y, Z are the solar coordinates or the geocentric, equatorial, rectangular coordinates of the Sun; ρ is the geocentric distance of the object, and the factors multiplying ρ are the direction cosines of the observation. These coordinates are expressed in "astronomical units" (a.u.) i.e. in units of the distance from the Earth to the Sun. This unit will be defined more precisely in the next chapter. The figure illustrates this coordinate system, with the position of the Earth corresponding to about November 1st, and a planet in right ascension ♈EF about 7^h, and declination PEF about +35°.



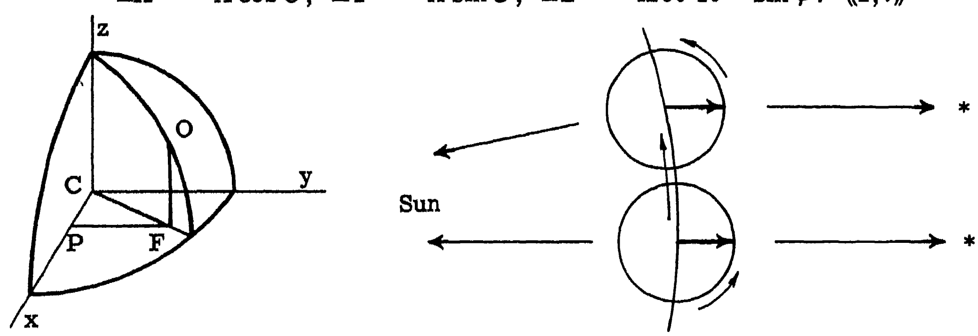
The solar coordinates which are needed at the time of an observation are obtained from their published ephemerides by interpolation. These are now printed with their first and second differences, and so Everett's formula is the most convenient one to use. A critical table of the 4-decimal values of the Everett 2nd difference coefficients is given in the appendix. It should be noted that the Everett 2nd difference coefficients are always negative, and because of the gravitational attraction, the second difference is of the opposite sign from the coordinate (except in some special cases in which the value of the coordinate is passing through zero). Therefore, the second difference effect is generally of the same sign as the coordinate; in other words, the true position lies farther from the origin than the position given by linear interpolation. The computer may avoid errors of sign by remembering to enter the 2nd difference products into the product dials of the computing machine with the same sign that the coordinate is entered.

Observations of the Sun itself show that it is actually slightly ahead of the position given by the theory from which the solar coordinates are computed. This discordance will be largely eliminated if the coordinates are interpolated, not for the time of observation, but for a time which is sufficiently later to allow the computed position of the Sun to move up to the actual position. This correction to the time of observation is $+0\overset{d}{.}000282$ for each second of arc required to correct the Sun's longitude. At present this correction is $+1\overset{''}{.}6$. A more elaborate correction for this discordance is given by Kahrstedt in the Astronomische Nachrichten, vol. 265, p. 313, but it is not necessary in ordinary cases.

Since the published ephemerides of the solar coordinates have the center of the Earth as the origin, whereas the observations are made from the Earth's surface, a slight correction for the parallax is required. This may be treated in either one of two ways. The first method adds to the solar coordinates the coordinates of the Earth's center referred to the observer's position on the surface at the time of the observation as origin. We thus have the coordinates of the Sun referred to the point of observation as origin, and the parallax is completely eliminated. The figure, at the left, represents one octant of the Earth, with the center at C, the observer at O; the xy-plane is the plane of the Earth's equator, and the positive z-axis is directed toward the north pole of the Earth's axis of rotation. The point F is the projection of O onto the xy-plane, and the point P is the projection of F onto the x-axis. The x-axis is directed toward the vernal equinox, therefore, the angle Θ = PCF is the local sidereal time.

Let ϕ' be the geocentric latitude, and express CO in astronomical units the same as the solar coordinates. Then $CF = A = 4266 \cdot 10^{-8} \cos\phi'$, and the topocentric corrections to be added to the solar coordinates are:

$$\Delta X = -A\cos\Theta, \quad \Delta Y = -A\sin\Theta, \quad \Delta Z = -4266\cdot 10^{-8}\sin\phi'. \qquad ((2,7))$$



The observations are usually not recorded in sidereal time, or "star time", but we may obtain the sidereal time from the mean solar time as follows. First consider an observer's local meridian at an instant when some star is in transit and it is exactly midnight, i.e. the Sun is on the opposite side of the Earth, as shown at the right in the figure. After one complete rotation of the Earth on its axis, the star (which is at an indefinitely great distance away) is again in transit; but due to the motion of the Earth in its orbit around the Sun, it is not yet again midnight. Almost a degree of rotation (or four minutes of time) is still needed to bring the Sun again exactly on the opposite side of the Earth. The period of time required for one complete rotation of the Earth on its axis is called a sidereal day. It is divided into 24 equal parts called sidereal hours, and each observer sets his own local sidereal time at 0^h when the vernal equinox crosses his meridian.

This leads to the simple rule that the right ascension of an object is equal to the sidereal time at which it transits the meridian.

In civil life we pay no attention to sidereal days, but count time in mean solar days. It is easy to see that because of the motion of the Earth around the Sun, the small difference each day accumulates to a total of one whole day in a year, and thus the number of solar days in a year is one less than the number of sidereal days or complete rotations of the Earth on its axis. We shall also see later that the rate at which the Earth revolves about the Sun is not constant, so that the additional amount of rotation of almost a degree each day is slightly different from day to day. Our ordinary clocks are regulated to run at a constant rate corresponding to the average number of solar days in a year, and this is known as mean solar time.

The relation between the sidereal time as determined by some stationary vernal equinox, such as 1950.0, and mean solar time may be expressed as $d\Theta_s = 1.002737803\,dt$. If the sidereal time is regulated by the slowly moving mean equinox of date, the relation is $d\Theta_m = 1.002737909\,dt$. If we find the value Θ_0 for the sidereal time on the meridian of Greenwich at some mean solar time, t_0, then $\Theta_s = \Theta_0 + L + 1.002737803\,(t - t_0)$, where L is the longitude of the observer on the Earth, measured around toward the east, $(t - t_0)$ is expressed in mean solar days and decimals of a day, and all the angles are expressed in decimals of a circle or units of 2π radians, sometimes called "gones". This subject is discussed and data concerning the principal observatories of the world are given in some volumes of the British Nautical Almanac and in the Lick Observatory Bulletin 445. The user must be careful to notice that the system of dates as given in the latter reference represents an attempt to pervert the established system of Julian Day Numbers by introducing a Julian Civil Time. No difficulty will be encountered if one reads JD 2419999.5 instead of 2420000.0 JCT. Also if one uses the θ given for each observatory, $L = \theta + 0.0897$, and finally

$$\Theta_s = \theta + 0.0014 + 1.002737803\,(JD - 2419999.5) = \Theta_0 + L + 1.002737803\,(t - t_0). \qquad ((2,8))$$

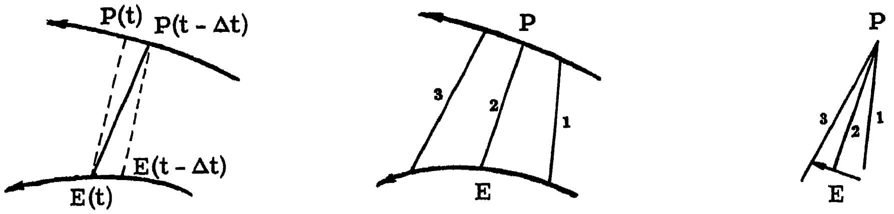
The second method is to be preferred when many observations are to be compared and a geocentric ephemeris has been computed. We are able to compute "parallax factors", which are the components of the displacement in space from the observer to the center of the Earth which are perpendicular to the line of sight and perpendicular and parallel, respectively, to the object's hour circle. When the geocentric distance of the object becomes known from the ephemeris, then the angular displacements in right ascension and declination are given by simple division, and these are applied to the observed values to give the corresponding geocentric values which would have been obtained if the object had been observed from the center of the Earth. The formulas may be found in Campbell, Elements of Practical Astronomy, New York, 1899, and they are given here without proof.

$$\rho p(\alpha) = 0^{s}\!.587\,(\rho'\cos\phi')\sin H\,\sec\delta \qquad \rho p(\delta) = 8''\!.80\,(\rho'\sin\phi')\sin(\gamma - \delta)\csc\gamma \qquad ((2,9))$$

where $\tan\gamma = \tan\phi'\sec H$, H is the local hour angle, and ρ' is the observer's geocentric distance.

Finally, we must consider the effect of the finite velocity of light upon the observations. It is evident that when the Earth has a component of motion perpendicular to the line of sight, the light does not reach the Earth along the line joining the Earth and the object, but along the line joining the Earth and the point where the object was at a previous time when the light left it. This aberration or "wandering" of the observed position from the actual, geometrical position at any given instant of time is easily seen to be a small angle whose tangent is the component of the Earth's velocity perpendicular to the line of sight divided by the speed of light. In the case of the stars, no attempt is made to determine the actual position of a star, but simply to determine the position in which the star would be seen if observed from the Sun, i.e. to correct the observed position for the motion of the Earth in its orbit. This kind of correction is known as the stellar or the annual aberration.

When the position of a planet or comet is observed with respect to neighboring stars, then both partake of the annual aberration and so it may be disregarded, but the light is still coming from the direction in which the object was at a previous time. Therefore, to make a comparison between an observed and a computed position of the object, the coordinates should be computed for a time which is earlier than the time of observation by this amount. At the left in the figure, the light which emanated in all directions from P at the time $(t - \Delta t)$ reaches E at the time t, and the



solid line is therefore the line of sight. The coordinates of E are computed for the time t, but the coordinates of P must be computed for the time $t - 0^{d}.00577\rho$, where the geocentric distance is expressed in astronomical units. This correction of the observed time is known as the light time.

Alternatively, suppose that an ephemeris has been computed by combining the geometrical positions of the Earth and the object at, say, 0^{h} of each day without regard for the effects of light time or aberration, i.e. the line EP(t). Then the relative positions on three successive days will be as shown in the center of the above figure. The positions of the Earth relative to the light source are shown at the right. The arrow indicates the component of the Earth's motion perpendicular to the line of sight; its magnitude is ρ times the "motion in two days". This is to be divided by the distance which light would travel in two days, in order to obtain the amount of the aberration. The distance which light would travel in two days is the number of seconds in two days divided by the number of seconds required to travel one unit of ρ. According to the figure, the apparent motion of the planet is retrograde but the relative motion of the Earth causes the observed position to be in advance of the computed position, therefore the correction to be applied to either coordinate of the original ephemeris in order to make a direct comparison with observation is

$$-0.0028841\,\rho\,(\text{motion in two days}).$$

The text will be illustrated with a complete example, using the observations of the minor planet (1361) Leuschneria = 1935 QA, which are published in the Astronomical Journal, vol. 45, p. 127. All the observations are there analyzed numerically for errors by testing the smoothness of their mean differences per day. This example should be studied by the student and the same process applied in similar actual cases. All these observations were made at Uccle.

At this stage we shall select five observations, reduce them to the equator and equinox of 1950.0, interpolate the solar coordinates, and correct them for parallax.

1935 Aug. 30.0006 UT	= JD 2428044.5006	$\alpha(1935.0) = 23^{h}\,05^{m}\,19^{s}.96$	$\delta(1935.0) = -3^{\circ}\,46'\,19''.7$
Sept. 2.9067	48.4067	23 02 55.72	-4 35 28.3
Sept. 6.9351	52.4351	23 00 22.96	-5 26 47.2
Sept. 23.8717	69.3717	22 50 15.46	-8 56 00.7
Oct. 21.8510	97.3510	22 42 49.41	-13 01 19.1

The computations for the first date are given in detail; those for the remaining dates are left as as exercise for the student. We obtain the necessary data from the British Nautical Almanac for 1935, Tables XIII, XIV, and pages 17, 51, 696.

$A_0 = +46^{s}.094$	$D_0 = +4'\,52''.25$	$\Delta\alpha = +46^{s}.40$	$\alpha(1950.0) = 23^{h}\,06^{m}\,06^{s}.36$
$A_1 = -\ 4.703$	$D_1 = \ -0.01$	$\Delta\delta \quad +4'\,52''.2+$	$\delta(1950.0) = -3^{\circ}\,41'\,27''.5-$
$A_2 = -\ 0.006$	$\tan\delta_0 = -0.06593$		

If we check this independently from Schorr's tables, we have

$\alpha = 23^{h}\,05^{m}\,43^{s}.16$	$m = +3^{s}.07313$	$\Delta\alpha = +46^{s}.40$	$\alpha(1950.0) = 23^{h}\,06^{m}\,06^{s}.36$
$\delta = -3^{\circ}\,43'\,53''.6$	$n\sin\alpha = -0.31352$	$\Delta\delta = +4'\,52''.3-$	$\delta(1950.0) = -3^{\circ}\,41'\,27''.4+$
$\tan\delta = -0.06522$	$n\cos\alpha = +19''.4837$		

In this case the check does not need to proceed by successive approximations because the tentative mean values of α and δ may now be formed with sufficient accuracy in advance.

The vernal equinox transits the meridian of Greenwich on 1935 Aug. 30.06300 UT. But the equinox of 1950.0 is 0.00056 in advance of this; and L = 0.01211 for Uccle. Therefore, at Uccle $\Theta = 0.01267 + 1.002737803\,(\text{JD} - 2428044.5630)$, and for Aug. 30.0006, $\Theta = 0.9501$, $\cos\Theta = 0.9513$, $\sin\Theta = -0.3084$, $A = 270$, $\Delta X = -257$, $\Delta Y = +83$, $\Delta Z = -329$, in units of 10^{-7}.

Before interpolating the solar coordinates, we add $0\overset{d}{.}00045$ to the time of observation. The Everett coefficients corresponding to n = 0.00105 are -0.0003, -0.0002. The interpolation for X, Y, and Z is as follows:

$$X = -0.9217058 - 66118(0.00105) + 2663(-0.0003) + 2686(-0.0002) - 257 = -0.9217386$$
$$Y = +0.3782831 - 144123(0.00105) - 1091(-0.0003) - 1048(-0.0002) + 83 = +0.3782763$$
$$Z = +0.1640664 - 62513(0.00105) - 471(-0.0003) - 453(-0.0002) - 329 = +0.1640270$$

The basic data for all of these observations is collected for future reference.

Date 1935	α(1950.0)	δ(1950.0)	Θ	X	Y	Z
Aug. 30.0006 UT	$23^h\ 06^m\ 06\overset{s}{.}36$	$-3°\ 41'\ 27\overset{''}{.}4$	0.9501	-0.9217386	+0.3782763	+0.1640270
33.9067	23 03 42.21	-4 30 36.8	0.8669	-0.9460249	+0.3214131	+0.1393582
37.9351	23 01 09.54	-5 21 56.5	0.9063	-0.9667071	+0.2612860	+0.1132835
54.8717	22 51 02.49	-8 51 13.7	0.8893	-1.0032412	-0.0014225	-0.0006612
82.8510	22 43 37.03	-12 56 35.2	0.9452	-0.8811272	-0.4245110	-0.1841615

CHAPTER 3

THE PROBLEM OF TWO BODIES

'Η δ' ἐπιστήμη ἐστὶ γλῶσσα τῶν θεῶν. Θεωρημίδης.

The problem of determining the path of an object in the solar system was described in the Introduction. The solution is obtained by substituting into the equation $F = ma$ the expression for the force given by the law of universal gravitation.

Consider two bodies of masses m and m_0, respectively, whose positions are referred to a system of rectangular coordinates (ξ, η, ζ) which is fixed in space or the so-called "unaccelerated axes" of classical mechanics. Let the distance between the bodies be ρ. The law of gravitation states that every particle of matter attracts (-) every other particle with a force that is directly proportional (k^2) to the product of their masses ($m m_0$) and is inversely proportional to the square ($1/\rho^2$) of the distance between them. The projection onto the ξ-axis of the gravitational forces acting between the two bodies gives

$$m\frac{d^2\xi}{dt^2} = -k^2\frac{m m_0(\xi - \xi_0)}{\rho^3} \qquad \text{and} \qquad m_0\frac{d^2\xi_0}{dt^2} = -k^2\frac{m m_0(\xi_0 - \xi)}{\rho^3} \qquad ((3,1))$$

and similarly for η and ζ, where k^2 is still to be determined. If we add the two equations of ((3,1)) and integrate twice, we obtain

$$m\frac{d^2\xi}{dt^2} + m_0\frac{d^2\xi_0}{dt^2} = 0, \qquad m\frac{d\xi}{dt} + m_0\frac{d\xi_0}{dt} = C_1, \qquad m\xi + m_0\xi_0 = C_1 t + C_2.$$

Thus we see that the center of mass moves uniformly in a straight line, and it may therefore be adopted as the origin of a system of unaccelerated axes. If the body m is acted upon by more than one other body, the equation ((3,1)) becomes

$$\frac{d^2\xi}{dt^2} = k^2\sum_i m_i\frac{(\xi_i - \xi)}{\rho_i^3}, \quad \frac{d^2\eta}{dt^2} = k^2\sum_i m_i\frac{(\eta_i - \eta)}{\rho_i^3}, \quad \frac{d^2\zeta}{dt^2} = k^2\sum_i m_i\frac{(\zeta_i - \zeta)}{\rho_i^3}, \qquad ((3,2))$$

where $\rho_i^2 = (\xi_i - \xi)^2 + (\eta_i - \eta)^2 + (\zeta_i - \zeta)^2$. These are the equations of motion when the origin of the coordinates is at the center of gravity or the barycenter of the system of bodies, and they may be integrated by the numerical methods of Chapter 1.

In the solar system, the Sun contains all but 1/700 of the total mass, and so it is a practical convenience to adopt the Sun as the origin of coordinates. Let the Sun be designated by $m_0 = 1$ and express all the other masses in this unit. Write $\xi - \xi_0 = x$, $\eta - \eta_0 = y$, $\zeta - \zeta_0 = z$, and for the Sun write r instead of ρ. Then $\frac{d^2\xi_0}{dt^2} = k^2\sum_i m_i\frac{(\xi_i - \xi_0)}{r_i^3}$, and subtracting this from the first of ((3,2)) gives:

$$\frac{d^2x}{dt^2} = -k^2(1+m)\frac{x}{r^3} + k^2\sum_i m_i\left(\frac{x_i - x}{\rho_i^3} - \frac{x_i}{r_i^3}\right) \qquad ((3,3))$$

and similarly for y and z. These equations of motion may also be solved by numerical integration and the work is greatly facilitated by the volumes of Planetary Coordinates. This procedure is known as Cowell's method, although this is not strictly accurate. Cowell's work may be consulted in the Monthly Notices of the Royal Astronomical Society, vol. 68, p. 576, and the appendix to the Greenwich Observations of 1909. Another valuable discussion of the subject by Jackson will be found in the M. N. R. A. S., vol. 84, p. 602. The details of this problem will be considered later.

It will greatly simplify the problem of finding a preliminary orbit, without seriously impair-

ing the accuracy of the results, if we set all the m_i's equal to zero in (3,3); and this will be done in that which follows. This is then known as the Problem of Two Bodies.

We now have three differential equations of the 2nd order, so that there will be six arbitrary constants in the solution, or six parameters are needed to represent all possible orbits. From (3,3), we have (with $m_i = 0$): $x\frac{d^2y}{dt^2} + \frac{dx}{dt}\frac{dy}{dt} - \frac{dx}{dt}\frac{dy}{dt} - y\frac{d^2x}{dt^2} = 0$, and similarly for a cyclical interchange of x, y, and z. If these three equations are integrated, we obtain

$$x\frac{dy}{dt} - y\frac{dx}{dt} = a_3, \qquad y\frac{dz}{dt} - z\frac{dy}{dt} = a_1, \qquad z\frac{dx}{dt} - x\frac{dz}{dt} = a_2, \qquad (3,4)$$

and from these equations we get $a_1x + a_2y + a_3z = 0$, so that the motion of the object is confined to a plane which passes through the Sun, and whose normal has the direction components a_1, a_2, a_3. Write $a_1^2 + a_2^2 + a_3^2 = c_1^2$, $a_1 = c_1 \sin i \sin\Omega$, $a_2 = c_1 \sin i \cos\Omega$, $a_3 = c_1 \cos i$. Then Ω is the longitude of the ascending node of the orbit plane upon the xy-plane and i is the inclination of the orbit plane to the xy-plane; these are two of the arbitrary constants, and they specify the position of the orbit plane in space.

With the orbit plane now supposed to be known, our problem is reduced to two dimensions and there remain four arbitrary constants to be determined. Referred to a new set of axes fixed in the orbit plane, we now have $a_1 = a_2 = 0$, $a_3 = c_1$, and the differential equations to be solved are:

$$\frac{d^2x}{dt^2} = -k^2(1+m)\frac{x}{r^3}, \qquad \frac{d^2y}{dt^2} = -k^2(1+m)\frac{y}{r^3}, \qquad (3,5)$$

where $r^2 = x^2 + y^2$. It is evident by inspection that if r were constant these equations would be satisfied by the sine and cosine. In the general case, the solution is complicated by the presence of this extraneous dependent variable in the equations. This difficulty may be circumvented by transforming from the rectangular coordinates, x and y, to the two independent variables in polar coordinates, r and v, by means of the transformation

$$x = r\cos v, \qquad y = r\sin v. \qquad (3,6)$$

Then

$$\frac{dx}{dt} = \cos v\frac{dr}{dt} - r\sin v\frac{dv}{dt}, \qquad \frac{dy}{dt} = \sin v\frac{dr}{dt} + r\cos v\frac{dv}{dt} \qquad (3,7)$$

and by substituting (3,6) and (3,7) into the first equation of (3,4), we obtain

$$c_1 = r\cos v\left(\sin v\frac{dr}{dt} + r\cos v\frac{dv}{dt}\right) - r\sin v\left(\cos v\frac{dr}{dt} - r\sin v\frac{dv}{dt}\right) = r^2\frac{dv}{dt} = 2\frac{dA}{dt}, \qquad (3,8)$$

where A is the area swept out by the radius vector. Integrating (3,8), we obtain

$$2A = c_1t + c_2, \qquad (3,9)$$

where c_2 is another one of the arbitrary constants.

The square of the instantaneous linear speed is given by

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{dv}{dt}\right)^2 = \left[\left(\frac{dr}{dv}\right)^2 + r^2\right]\left[\frac{dv}{dt}\right]^2 = \left[\left(\frac{dr}{dv}\right)^2 + r^2\right]\frac{c_1^2}{r^4} \qquad (3,10)$$

The derivative of the left hand member is $2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2}$ and in this expression we not only may substitute from (3,7), but we may now also impose the conditions contained in our basic differential equations (3,5). Thus

$$-2\left(\cos v\frac{dr}{dt} - r\sin v\frac{dv}{dt}\right)\frac{k^2(1+m)\,r\cos v}{r^3} - 2\left(\sin v\frac{dr}{dt} + r\cos v\frac{dv}{dt}\right)\frac{k^2(1+m)\,r\sin v}{r^3} = -2\frac{k^2(1+m)}{r^2}\frac{dr}{dt}$$

The integral of this expression is then equal to the right hand member of (3,10), except for an arbitrary constant of integration. Thus $\frac{2k^2(1+m)}{r} + c_3 = \left[\left(\frac{dr}{dv}\right)^2 + r^2\right]\frac{c_1^2}{r^4}$, or if we separate dr and dv,

$$\sqrt{\frac{2k^2(1+m)}{r} + c_3 - \frac{c_1^2}{r^2}}\,dv = \frac{c_1}{r^2}dr \quad \text{or} \quad dv = \frac{-c_1 d(1/r)}{\sqrt{c_3 + \frac{k^4(1+m)^2}{c_1^2} - \left(\frac{c_1}{r} - \frac{k^2(1+m)}{c_1}\right)^2}} = \frac{-du}{\sqrt{B^2 - u^2}} \qquad (3,11)$$

where $u = \frac{c_1}{r} - \frac{k^2(1+m)}{c_1}$, and $B^2 = c_3 + \frac{k^4(1+m)^2}{c_1^2}$. Integrating ((3,11)), we obtain

$$v = \arccos\frac{u}{B} + c_4 \quad \text{or} \quad \cos(v - c_4) = \frac{u}{B} = \frac{\frac{c_1}{r} - \frac{k^2(1+m)}{c_1}}{\sqrt{c_3 + \frac{k^4(1+m)^2}{c_1^2}}}$$

and from this we obtain

$$r = \frac{p}{1 + e\cos(v - c_4)} \qquad ((3,12))$$

where $p = \frac{c_1^2}{k^2(1+m)}$ and $e = \sqrt{1 + \frac{c_3c_1^2}{k^4(1+m)^2}}$. Now ((3,12)) is the equation of a conic section in polar coordinates; p is the parameter of the conic or the semi-latus rectum, and e is the eccentricity. If $e<1$, then $c_3 = -\frac{k^2(1+m)}{a}$, where $a = p(1 - e^2)$ is the semi-major axis of the ellipse. Then ((3,10)) and the equation preceding ((3,11)) give us the important equation

$$G^2 = \frac{2}{r} - \frac{1}{a} \qquad ((3,13))$$

where G is the linear speed in the orbit in units of $k\sqrt{1+m}$ mean solar days. This equation ((3,13)) expresses the law of conservation of energy for the system. If we multiply both sides by $\frac{1}{2}m$ and transpose: $\frac{1}{2}mG^2 - \frac{m}{r} = -\frac{m}{2a}$, i.e. the sum of the kinetic and potential energy is a constant.

If $e>1$, we have a hyperbolic orbit. In order to avoid imaginaries, we change the sign of a in the definition and write $a = p(e^2 - 1)$. This gives the equation $G^2 = 2/r + 1/a$ for the hyperbola instead of ((3,13)). The parabola is the limiting case in which $G^2 = 2/r$.

The constants of integration are now all determined: $c_1 = k\sqrt{p(1+m)}$, $c_3 = \frac{k^2(1+m)(e^2 - 1)}{p}$, c_2 determines the amount of area already swept out at $t=0$, or the position of the body in its orbit at the origin of time, and if v is counted from the perihelion, then $c_4 = \omega$, the argument of perihelion or the angle measured along the orbit plane from the ascending node to the perihelion of the conic section. The preceding analysis has been patterned after Moulton, Celestial Mechanics, New York, 1914, chap. V. Later we shall see that other sets of six constants may also be used to define the orbit.

We may now also determine the value of k. At this point we are confronted with the interconnection between purely mathematical theory and the actual physical processes of material particles, which can be determined only by observation. We shall employ the observed value of the period of revolution of the Earth, expressed in mean solar days, and the mass of the Earth, expressed in units of the Sun's mass, as determined from observations of the perturbative action of the Earth on other objects, principally the Moon.

Let $t_2 - t_1 = P$, one complete period of revolution of the Earth, and then from the integral of ((3,8)), and noticing that $(A_2 - A_1)$ is the whole area of the ellipse, we have

$$2(A_2 - A_1) = c_1P = Pk\sqrt{(1+m)a(1-e^2)} = 2\pi a^2\sqrt{1-e^2} \quad \text{or} \quad k = \frac{2\pi a^{3/2}}{P\sqrt{1+m}} \qquad ((3,14))$$

If we agree to measure all distances in astronomical units, such that $a = 1$, then with the values of P and m used by Gauss, one obtains $k = 0.01720209895$ (per mean solar day). This is known as the Gaussian constant. It would be inconvenient to change k every time better determinations of P and m are obtained, so the value of k is held fixed by common consent, and then ((3,14)) gives the value of a for the Earth's orbit in terms of the adopted fictitious unit of distance. This unit of distance is the radius which the orbit of a massless particle would have if it travelled around the Sun in a circle at the rate of k radians per mean solar day.

The use of vector notation not only will simplify the analysis but especially in orbit work it gives a clear geometrical picture of the meaning and effect of the various operations that are performed. We shall therefore consider the elementary notions of vector analysis, and for the benefit of illustration and comparison, repeat the preceding proofs.

A vector is a segment of a straight line which has both length and direction. A point P whose coordinates are x, y, z may be regarded as being specified by the position vector $\mathbf{r}$ extending from the origin to the point P. It is usual to denote by $\mathbf{i}$ a unit vector directed from the origin to the point (+1,0,0), by $\mathbf{j}$ a unit vector from the origin to (0,+1,0), and by $\mathbf{k}$ a unit vector to (0, 0, +1). Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and the absolute magnitude or length of $\mathbf{r}$ is $r = \sqrt{x^2 + y^2 + z^2}$. The sum of two vectors is given by $\mathbf{r}_1 + \mathbf{r}_2 = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}$. The subtraction of a vector simply reverses its direction. A graphical representation shows that vectors are added according to the "parallelogram rule".

The scalar or "dot" product of two vectors is defined as

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos(\mathbf{r}_1, \mathbf{r}_2) = x_1 x_2 + y_1 y_2 + z_1 z_2 \qquad ((3,15))$$

The dot product is obviously commutative, and it is zero if the two vectors are perpendicular to each other. The following relations occur frequently: $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$, $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$. Also $x = \mathbf{r}\cdot\mathbf{i}$, $y = \mathbf{r}\cdot\mathbf{j}$, $z = \mathbf{r}\cdot\mathbf{k}$, and $(\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v}) = u^2 + v^2 - 2(\mathbf{u}\cdot\mathbf{v})$. This is the "law of cosines" from plane trigonometry.

The vector or "cross" product of two vectors is defined as another vector whose magnitude is $r_1 r_2 \sin(\mathbf{r}_1, \mathbf{r}_2)$ and which is directed perpendicular to the plane of $\mathbf{r}_1$ and $\mathbf{r}_2$ in the direction of a right-handed set, or

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & x_1 & x_2 \\ \mathbf{j} & y_1 & y_2 \\ \mathbf{k} & z_1 & z_2 \end{vmatrix} \qquad ((3,16))$$

The cross product is zero if the two vectors are parallel, and $\mathbf{r}_1 \times \mathbf{r}_2 = -\mathbf{r}_2 \times \mathbf{r}_1$. Also $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$. The absolute magnitude of the vector product is double the area of the triangle formed upon the two vectors as sides.

The triple scalar product, $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$, is obtained in terms of the previous definitions by first performing the cross product and then the dot product, or

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \begin{vmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{vmatrix} \qquad ((3,17))$$

From this it is evident that the result is not affected by an interchange of the dot and the cross or by a cyclical interchange of the arrangement of the vectors, but an interchange of any two of them changes the sign of the result. The value of the triple scalar product is double the volume of the tetrahedron formed upon the three vectors as edges.

The triple vector product $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \mathbf{V}$ is a vector which is perpendicular to both $\mathbf{u}\times\mathbf{v}$ and $\mathbf{w}$. Since $\mathbf{u}\times\mathbf{v}$ is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{V}$ must be coplanar with $\mathbf{u}$ and $\mathbf{v}$. This may be expressed as $\mathbf{V} = m\mathbf{u} + n\mathbf{v}$. Also since $\mathbf{V}$ is perpendicular to $\mathbf{w}$, their dot product is zero, i.e. $m(\mathbf{u}\cdot\mathbf{w}) + n(\mathbf{v}\cdot\mathbf{w}) = 0$. If we substitute $m = q(\mathbf{v}\cdot\mathbf{w})$ and $n = -q(\mathbf{u}\cdot\mathbf{w})$, then we have only to find q, a factor of proportionality, in order to have the complete expression for $\mathbf{V}$. We may do this with no loss of generality if we write: $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, $\mathbf{w} = c\mathbf{i} + d\mathbf{j} + e\mathbf{k}$. Then $\mathbf{u}\times\mathbf{v} = b\mathbf{k}$, and $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = -bd\mathbf{i} + bc\mathbf{j} = q(ac + bd)\mathbf{i} - q\,c(a\mathbf{i} + b\mathbf{j})$. By equating coefficients, we see that $q = -1$. Thus

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}. \qquad ((3,18))$$

If $\mathbf{w}$ is designated as the "outer" vector because it is outside ( ), $\mathbf{v}$ as the "adjacent" vector and $\mathbf{u}$ as the "remote", because of their positions with respect to $\mathbf{w}$, then ((3,18)) may be remembered by the following mnemonic, "(Outer dot Remote) Adjacent minus (Outer dot Adjacent) Remote."

If the point P is moving along some space curve, then the position vector $\mathbf{r}$ is a variable, and the derivative of $\mathbf{r}$ with respect to t or the velocity vector is given by

$$\mathbf{r}' = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

Then also $\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$, and all the ordinary processes of calculus are also applicable to vectors. We may notice that when $\mathbf{r}^*$ is a variable unit vector rotating in a plane, then $\frac{d\mathbf{r}^*}{dt} = V\mathbf{T}$,

where $\mathbf{T}$ is a unit vector tangent to the circle which $\mathbf{r}^*$ describes, and $V = \frac{d\theta}{dt}$. Thus we have

$$\mathbf{r}^* \cdot \mathbf{T} = 0 \tag{(3,19)}$$

As a familiar illustration of the application of vectors, we shall show how the entire subject of spherical trigonometry may be derived from two vector expressions. Consider a sphere of unit radius and three unit vectors, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, directed from its center to the three vertices, A, B, C, respectively, of the spherical triangle on its surface. Let the side opposite each of the vertices be denoted by a, b, c, respectively. The expression

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C})$$

may be regarded as a triple scalar product in which $(\mathbf{A} \times \mathbf{B})$ is a single vector. Interchange the dot and the cross, and then expand the resulting triple vector product.

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{A} \cdot \mathbf{C} = [(\mathbf{A} \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{A}] \cdot \mathbf{C} = (\mathbf{B} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})(\mathbf{A} \cdot \mathbf{C}) \tag{(3,20)}$$

Also $\mathbf{B} \cdot \mathbf{C} = \cos a$, $\mathbf{A} \cdot \mathbf{B} = \cos c$, $\mathbf{A} \cdot \mathbf{C} = \cos b$, $\mathbf{A} \times \mathbf{B} = \sin c\, \mathbf{U}_1$, $\mathbf{A} \times \mathbf{C} = \sin b\, \mathbf{U}_2$, and $\mathbf{U}_1 \cdot \mathbf{U}_2 = \cos A$. Making these substitutions in ((3,20)) and transposing gives the "law of cosines" for spherical trigonometry

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly the expression

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{A} \times \mathbf{C})$$

may be regarded as a triple vector product in which $(\mathbf{A} \times \mathbf{B})$ is the "outer" vector. Then

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{A} \times \mathbf{C}) = [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}]\mathbf{A} - [(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{A}]\mathbf{C} = K\mathbf{A} \tag{(3,21)}$$

where K is the triple scalar product $[\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}]$. The left hand member of this equation becomes $\sin c \sin b\, \mathbf{U}_1 \times \mathbf{U}_2 = \sin c \sin b \sin A\, \mathbf{A}$; therefore (assuming a cyclical interchange of the vectors) the absolute magnitude of ((3,21)) gives

$$K = \sin c \sin b \sin A = \sin a \sin c \sin B = \sin b \sin a \sin C$$

and if we divide through by $\sin a \sin b \sin c$, we obtain the "law of sines".

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

All the other formulas of spherical trigonometry may be obtained from these two "laws" by substitution and transposition. A more detailed treatment of the algebra and calculus of vectors, with numerous applications and examples, will be found in Brand, Vectorial Mechanics, New York, 1930.

Returning now to our fundamental equation $F = m\,a$, we notice that this is inherently a vector equation and, similar to ((3,1)), we have

$$m\frac{d^2\mathbf{r}}{dt^2} = -k^2\frac{mM}{r^2}\mathbf{r}^* \tag{(3,22)}$$

where $\mathbf{r}^*$ is a unit vector directed outward along the radius. Operate upon both sides of this equation by $\mathbf{r}\times$, and then integrate:

$$\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = 0 \quad \text{and} \quad \mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{h} \tag{(3,23)}$$

where $\mathbf{h}$ is a vectorial constant of integration. Therefore $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}$ always lie in a fixed plane whose normal is the constant vector $\mathbf{h}$, and the rate at which area is swept out by $\mathbf{r}$ is constant. These properties are not dependent upon the law of gravitation; they are true for any central force, either attractive or repulsive. This may also be seen to be true by considering $\mathbf{I}$, the angular momentum vector of the system. Write $\mathbf{I} = \mathbf{r} \times m\mathbf{v}$; then $\frac{d\mathbf{I}}{dt} = \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} = 0 + \mathbf{r} \times \mathbf{F}$, and if we have any central force directed along $\mathbf{r}$, then the last cross product also becomes zero, and $\mathbf{I}$ is a constant.

If we write $\mathbf{r} = r\,\mathbf{r}^*$, then $\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{r}^* + r\frac{d\mathbf{r}^*}{dt}$, and the substitution of both of these expressions into ((3,23)) gives

$$\mathbf{h} = r^2\,\mathbf{r}^* \times \frac{d\mathbf{r}^*}{dt} \qquad ((3,24))$$

Now cross the members of ((3,22)) by the members of ((3,24)):

$$\frac{d^2\mathbf{r}}{dt^2} \times \mathbf{h} = -\frac{k^2M}{r^2}\mathbf{r}^* \times \left(r^2\,\mathbf{r}^* \times \frac{d\mathbf{r}^*}{dt}\right) = k^2M\frac{d\mathbf{r}^*}{dt}. \qquad ((3,25))$$

In expanding this triple vector product, we must notice that $\mathbf{r}^*$ is a unit vector rotating in the orbit plane, and according to ((3,19)) its dot product with its own derivative is zero. Integrate ((3,25)):

$$\frac{d\mathbf{r}}{dt} \times \mathbf{h} = k^2M(\mathbf{r}^* + \mathbf{e}) \qquad ((3,26))$$

where $\mathbf{e}$ is the second and final vectorial constant of integration. If we operate upon both sides of ((3,26)) by $\mathbf{r}\cdot$, we obtain

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} \times \mathbf{h} = \mathbf{r} \times \frac{d\mathbf{r}}{dt}\cdot\mathbf{h} = h^2 = k^2M\,\mathbf{r}\cdot(\mathbf{r}^* + \mathbf{e}) = k^2M\,r\,(1 + e\cos v)$$

or

$$r = \frac{p}{1 + e\cos v} \qquad ((3,27))$$

which is the equation of a conic section in polar coordinates, where e is the eccentricity of the conic, $h^2/k^2M = p$ is the parameter or the semi-latus rectum (the value of r when $v = 90°$), and the substitution of $\mathbf{r}\cdot\mathbf{e} = r\,e\cos v$ means that the angle v is counted from $\mathbf{e}$ as the initial line.

We may observe that $\mathbf{h}$ is normal to the orbit plane and its absolute magnitude is $\sqrt{p}$, if we measure mass in units of the Sun's mass, distance in astronomical units, and time in units of $1/k$ mean solar days. Thus $\mathbf{h}$ corresponds to three of the scalar constants of integration: i, Ω, and p. The other constant of integration, $\mathbf{e}$, also corresponds to three of the scalar constants, although only two of these are evident because we have projected the equation ((3,26)) onto $\mathbf{r}$. Thus $\mathbf{e}$ has an absolute magnitude equal to the eccentricity, and it is directed toward the perihelion, giving e and ω. But if the projection of ((3,26)) onto $\mathbf{r}$ has dissolved any of the information which the equation originally contained, this information will be exhibited if we operate on the equation with $\times\,\mathbf{r}$. Thus we obtain $(\mathbf{r}' \times \mathbf{h}) \times \mathbf{r} = (\mathbf{r}\cdot\mathbf{r}')\mathbf{h} = k^2M\,\mathbf{e} \times \mathbf{r}$ or if we dot through by $\mathbf{h}$, we have the scalar equation $(\mathbf{r}\cdot\mathbf{r}')p = (\mathbf{e} \times \mathbf{r}\cdot\mathbf{h}) = \sqrt{p}\,r\,e\sin v$, which indicates the position of the body in its orbit and permits the determination of T, the time of perihelion passage.

The preceding demonstrations have been based upon Newton's laws of motion and gravitation. On this basis we are led to the proofs of three laws of planetary motion which were originally discovered empirically by Kepler through his geometrical analysis of the observations of Mars which had been made by Tycho Brahe. It is interesting to speculate upon the probable development of this aspect of science if the orbit of Mars had not happened to be of such a large eccentricity as to enable Kepler to distinguish the properties he discovered. Kepler's laws may be stated as follows:

> Each planet moves about the Sun in an ellipse (in fact, more generally, in a conic section) with the Sun in one focus.
>
> The radius vector sweeps over equal areas in equal intervals of time.
>
> The squares of the periods of revolution of the planets about the Sun are proportional to the cubes of their mean distances from the Sun.

This last law is seen to be true by observing that ((3,14)) is as equally applicable to any other planet as to the Earth. The law is in error to the extent that the masses of the planets are not equal, and that the planets are not all equally affected by their mutual gravitational action, which in the Two Body Problem has been neglected by setting $m_1 = 0$.

Summarizing the results we have obtained thus far in the Two Body Problem, we have found that the one body is constrained to move in a conic section relative to the other body, such that the second body remains in one focus, and the path may be defined by the following constants of

integration or elements of the orbit, usually referred to the ecliptic (the plane of the Earth's orbit) as the fundamental plane of reference.

The longitude of the node, Ω, is the angle measured eastward along the ecliptic from the vernal equinox to the line of intersection with the orbit plane at which the motion in the orbit is from south to north, i.e. the ascending node.

The inclination, i, is the dihedral angle between the orbit plane and the ecliptic. It varies from 0° to 90° for direct or eastward motion about the Sun, and from 90° to 180° for retrograde motion.

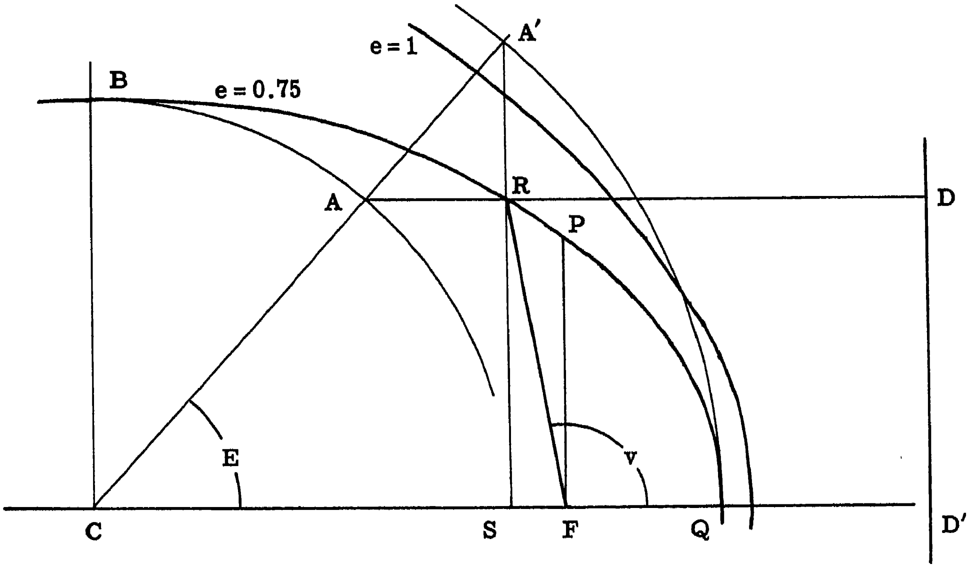
The argument of the perihelion, ω, is the angle measured in the orbit plane and in the direction of motion, from the ascending node to the perihelion.

The eccentricity, e, defines the shape of the conic section.

The mean distance, a, is the semi-major axis of the ellipse. For a hyperbolic orbit this becomes a negative quantity. For a parabolic orbit it is undefined, and it is replaced by q, the perihelion distance or the distance from the Sun to the object when the latter is at perihelion or nearest the Sun.

The time of perihelion passage, T, is usually given instead of some quantity associated with c_2. In an elliptic orbit, the mean anomaly, M_0, at some epoch, t_0, may be given instead of T.

It now remains to determine the position of the object in its orbit due to its motion during the time $(t - T)$ or $(t - t_0)$. Before investigating this problem, we shall review briefly some of the simple properties of conic sections. Given a fixed point F, a fixed line DD′, and a variable point R, such that the ratio of the distances from R to F and to DD′ is a constant, then the locus of R is a conic section with one focus at F and with eccentricity $e = RF/RD$. The adjoining figure shows



the first quadrant of an ellipse, with its major and minor auxiliary circles. The center of the ellipse is at C, the radius vector is $r = FR$, the mean distance or semi-major axis is $a = CQ = BF$, the semi-minor axis is $b = CB$, the semi-latus rectum is $p = FP$, the angle of eccentricity is $\phi = CBF = \arcsin e$, and the perihelion distance is $q = FQ$. It can be shown that $CF = ae$ and $a = \frac{b}{\sqrt{1 - e^2}} = \frac{p}{1 - e^2} = \frac{q}{1 - e}$. The angle RFQ is called the true anomaly; it is usually designated by v and it is measured from the perihelion, positively in the direction of motion. The angle ACQ is called the eccentric anomaly; it is usually designated by E and it vanishes with v. When A is the intersection of CA with the minor auxiliary circle of radius b, and A′ is the intersection of the extension of CA with the major auxiliary circle of radius a, then if RA is parallel to CF and RA′ is perpendicular to CF, the locus of R is also the ellipse. Furthermore, $SF = r \cos v = a(\cos E - e)$

and $SR = r\sin v = a\sqrt{1-e^2}\sin E$. As the eccentricity approaches unity, the point Q moves slightly to the right and the point C moves without limit to the left. When $e = 1$, the locus of R is a parabola and $p = 2q$. If e is greater than unity, the locus of R is a hyperbola, but we shall not be concerned with this case since strongly hyperbolic orbits have never been encountered in the solar system.

We return now to the equation ((3,8)), and substitute the value we have found for c_1.

$$r^2\frac{dv}{dt} = k\sqrt{p} \qquad ((3,28))$$

We shall drop the factor $\sqrt{1+m}$ which is always associated with k; in the cases of minor planets or comets, m is set equal to zero because it is unobservably small.

If the orbit is parabolic, ((3,27)) may be transformed to $r = \frac{2q}{1+\cos v} = q\sec^2\frac{1}{2}v$, and ((3,28)) becomes

$$\frac{k\,dt}{\sqrt{2}\,q^{3/2}} = \left(\sec^2\tfrac{1}{2}v + \sec^2\tfrac{1}{2}v\tan^2\tfrac{1}{2}v\right)d(\tfrac{1}{2}v)$$

If this equation is integrated from a lower limit T on the left hand side, corresponding to $v = 0$ on the right hand side, to a variable upper limit, we have

$$\frac{k(t-T)}{\sqrt{2}\,q^{3/2}} = \tan\tfrac{1}{2}v + \frac{1}{3}\tan^3\tfrac{1}{2}v \qquad ((3,29))$$

The solution of this equation when t is given has been tabulated in what is known as Barker's Table, but the solution is also readily obtained without recourse to tables in the following manner. When the time interval $(t - T)$ is given, let the left hand member of ((3,29)) be designated by N, and let $\tan\frac{1}{2}v = x$. Write the equation ((3,29)) in the form

$$f(x) = x + \frac{1}{3}x^3 - N = 0.$$

Then $f'(x) = 1 + x^2$, and, by Newton's method of approximation,

$$x_{i+1} = x_i + \Delta x = x_i - \frac{x_i + x_i^3/3 - N}{1+x_i^2} = \frac{N + 2x_i^3/3}{1+x_i^2}. \qquad ((3,30))$$

This last expression neglects $(\Delta x)^2$ and therefore it must be applied repeatedly until the successive solutions converge to their final value. The more accurately x_i is estimated at the beginning, the more rapidly the successive solutions will converge. As an example, consider the case in which $N = 2.0$, and for the sake of the illustration we deliberately assume $x_i = 1$. Then the successive solutions for x are: 1.3333 3333, 1.2888 8889, 1.2879 1022, 1.2879 0975.

When the orbit is not a parabola, we may write ((3,28)) in the form $\frac{k\,dt}{p^{3/2}} = \frac{dv}{(1+e\cos v)^2}$. The integral of this equation is $\frac{k(t-T)}{p^{3/2}} = \int_0^v (1+e\cos v)^{-2}dv = v - 2e\sin v + \ldots$, but this form of the solution is, in general, neither practicable nor useful.

It may be observed in the preceding figure that

$$r^2 = \overline{RS}^2 + \overline{SF}^2 = a^2(1-e^2)\sin^2 E + a^2(\cos E - e)^2 = a^2(1 - e\cos E)^2.$$

Therefore, if we write $$r = a(1 - e\cos E) = p(1 + e\cos v)^{-1},$$

we obtain $$dr = ae\sin E\,dE = pe\sin v\,(1+e\cos v)^{-2}dv = \frac{r^2 e\sin v}{p}dv$$

and ((3,28)) becomes $$r^2dv = k\sqrt{p}\,dt = ap\frac{\sin E}{\sin v}dE = \frac{rp}{\sqrt{1-e^2}}dE$$

or $$(1 - e\cos E)\,dE = \frac{k}{a^{3/2}}dt = n\,dt$$

The integral of this expression is known as Kepler's Equation:

$$M = n(t-T) = E - e\sin E \qquad ((3,31))$$

The M in this equation is known as the mean anomaly. It will be observed by referring back to ((3,14)) that $n = \frac{k}{a^{3/2}} = \frac{2\pi}{P}$, so that n is the mean motion per unit of time. Hence, we may write $M = M_0 + n(t - t_0)$, where M_0 is the mean anomaly at the epoch t_0; and thus in an elliptic orbit the element T may be replaced by M_0 at a given epoch t_0.

Literally hundreds of methods have been given for the solution of Kepler's equation. With a modern calculating machine and a table of sines having the argument expressed in decimals, the following method is the most efficient: write the equation in the form $E = M + e \sin E$, where e is expressed in the same units as M. Put M into the product counter of the machine and then set e on the keyboard. This is to be multiplied by such a number as will become the sine of the angle which results in the product counter. This process requires some judicious juggling, but it is not difficult after a little practice. Some computers may prefer to use a second machine to calculate the interpolation of sin E from the table of sines, but if the table is sufficiently extensive the closest tabular entry for E can be found by a little testing with the multiplier bar, and then the interpolation is made either mentally or with a slide rule. An excellent table for this purpose is Peters, Siebenstellige Werte der Trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades, Berlin, 1918.

For purposes of illustration, the student may compute the solution for the following values: M = 18°.11127, 19°.73041, 21°.34954, 22°.96867, using e = 11°.11916 = 0.1940659. The process for the first case is given in detail. The successive approximations to E are shown in the first column, the corresponding approximate values of sin E which are built up in the multiplier dials are shown in the second column, and the resulting values of E which appear in the product dials are shown in the third column. The solutions for the four given values of M are shown in the fourth column. From the run of the differences, it would be possible to estimate the next solution very closely, thus eliminating the need for the first few approximations.

E	sin E	E				
18°.1	0.31	21°.55820				
21.6	0.37	22.22535	22°.33718			
22.2	0.38	22.33654	24.30742	1.97024	657	
22.336	0.3800374	22.33696	26.27109	1.96367	697	40
22.337	0.3800536	22.33714	28.22779	1.95670		
22.33715	0.3800560	22.33716	(30.177)			
22.33718	0.3800565	22.33718				

A method of iteration may be employed for the solution of ((3,31)) by writing

$$E - M = e \sin E = A = (e \sin M)\cos A + (e \cos M)\sin A.$$

Start with $A = 0$ on the right hand side, and solve for A by repeated substitutions until the solution converges to its final value.

The following method is useful when an extensive table of sines and cosines is not available. Find the closest value of E_0 for which $\sin E_0$ and $\cos E_0$ are known. Let $M_0 = E_0 - e \sin E_0$, $D = E - E_0$ and write

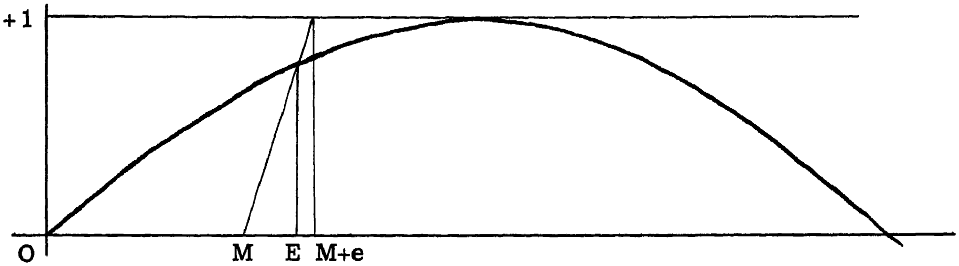
$$\begin{aligned} M - M_0 &= D - e[\sin(E_0 + D) - \sin E_0] \\ &= D - eD[\cos E_0(1 - D^2/6 + \ldots) - \sin E_0(D/2 - D^3/24 + \ldots)] \\ &= D - eDS \end{aligned}$$

and finally $D = \frac{M - M_0}{1 - eS}$, where S is obtained by repeated substitutions of D. If M and D are both expressed in degrees and decimals, then

$$S = \cos E_0 [1.0 - 0.0000\,5077\,(D^\circ)^2 + \ldots] - \sin E_0 [+0.0087\,2665\,(D^\circ) - 0.0000\,0022\,(D^\circ)^3 + \ldots].$$

This method is described by Draper in the Astronomical Journal, vol. 42, p. 123.

If we write $S = \cos E_0$, we have the equivalent of Newton's method of approximation for the solution of Kepler's equation, namely: $E - E_0 = (M - M_0)/(1 - e \cos E_0)$. The graphical solution of Kepler's equation is almost self-evident from the figure. On a graph of $y = \sin E$, lay off M on the E-axis and erect a line whose slope is $1/e$, i.e. the line $y = (E - M)/e$. Then the intersection of the line and the sine curve gives the solution for E, i.e. by eliminating y from the two equations we obtain the condition $M = E - e \sin E$ at the intersection.



In a hyperbolic orbit an analogous equation exists. In this case, a(1 - e) becomes negative, but by definition we agree to change the sign of a wherever it formerly appeared and to write instead a(e - 1). Let

$$r = a(e\cosh F - 1) = p(1 + e\cos v)^{-1}.$$

Then

$$dr = a e \sinh F\, dF = \frac{a(e^2 - 1)\, e \sin v}{(1 + e\cos v)^2} dv = \frac{r^2 e \sin v}{a(e^2 - 1)} dv$$

This time ((3,28)) becomes $r^2 dv = k\sqrt{a^2(e^2 - 1)}\, dt = a^2(e^2 - 1)\dfrac{\sinh F}{\sin v} dF = r a\sqrt{e^2 - 1}\, dF$

or

$$(e\cosh F - 1)\, dF = \frac{k\, dt}{a^{3/2}} = \nu dt$$

The integral of this equation is

$$\nu(t - T) = -F + e\sinh F. \qquad ((3,32))$$

But the only actual cases of hyperbolic orbits that are found in the solar system are orbits which have eccentricities so nearly equal to unity that this equation tends to become an indeterminate form. The same is true of Kepler's equation as the eccentricity approaches unity. These are then designated as "nearly-parabolic" orbits, and they require special treatment.

Taking a cue from ((3,29)), let $x = \tan\frac{1}{2}v$ and $u = \frac{1 - e}{1 + e}x^2$. Then $dx = \sec^2\frac{1}{2}v\, d(\frac{1}{2}v)$, $dv = \frac{2\, dx}{1 + x^2}$, and $r = \frac{q(1 + x^2)}{1 + u}$. Then equation ((3,28)) becomes

$$\frac{\sqrt{1 + e}\, k\, dt}{2\, q^{3/2}} = \frac{1 + x^2}{(1 + u)^2} dx$$

If the expression on the right is expanded term by term, and this equation is then integrated in the same manner as ((3,29)), we obtain

$$\frac{\sqrt{1 + e}\, k(t - T)}{2\, q^{3/2}} = x(1 - \frac{2}{3}u + \frac{3}{5}u^2 - \ldots) + \frac{1}{3}x^3(1 - \frac{6}{5}u + \frac{9}{7}u^2 - \ldots) \qquad ((3,33))$$

This form of solution suffers from the fact that beyond v = 90° the powers of u increase rapidly in value and a larger and larger number of terms must be taken into account. This is especially bad in orbits of small perihelion distance, for then the comet may be observed to large values of the true anomaly.

The most elegant and practical method of dealing with nearly parabolic orbits is one devised by Gauss. We shall treat this topic at greater length than is usual because it illustrates a very unfortunate situation which exists all too often so far as the education of the student is concerned. Instead of presenting material such as this in the manner in which it was discovered, including all the pitfalls and futile attempts, it is almost invariably set down in a polished form which bears no resemblance to its origination. The student, instead of being privileged to experience, even vicariously, the thrill of discovery and to profit from a successful mathematical conquest, is plunged, blindfolded, into the midst of a sea of results. An attempt has been made to reconstruct the course of events in this case, even though we have no more basis of information than is given in Gauss' Theoria Motus Corporum Coelestium. Furthermore, this problem is of unusual interest because it presents a rare, actual application in which circular and hyperbolic functions have a continuous relationship to each other as the eccentricity varies across unity; and the results may be easily visualized.

We set down in juxtaposition, for the sake of easy comparison, the equation ((3,29)) for the parabola, the previous equation ((3,33)) for the nearly parabolic orbit, and Kepler's equation ((3,31)) in a modified form.

$$\begin{aligned} \frac{k(t-T)}{\sqrt{2}\,q^{3/2}} &= x + \frac{1}{3}x^3 \\ \sqrt{\frac{1+e}{2}}\,\frac{k(t-T)}{\sqrt{2}\,q^{3/2}} &= x\left(1 - \frac{2}{3}u + \frac{3}{5}u^2 - \ldots\right) \\ &\quad + \frac{1}{3}x^3\left(1 - \frac{6}{5}u + \frac{9}{7}u^2 - \ldots\right) \\ \frac{k(t-T)}{q^{3/2}} &= \frac{E - e\sin E}{(1-e)^{3/2}} \end{aligned} \qquad ((3,34))$$

It will be observed that as e approaches unity the second equation reduces to the first, as indeed it should, and the last equation becomes an indeterminate form, 0/0. The deviation of the second equation from the first one is dependent not only upon the eccentricity (this effect is contained in the left hand factor of u and the extra factor on the left hand side of the equation) but also upon the angle from perihelion. Thus the factors in parentheses which multiply the principal terms on the right hand side reduce to unity at perihelion, irrespective of the eccentricity. If we attempt to obtain an expression of the form

$$F_1 \frac{k(t-T)}{\sqrt{2}\,q^{3/2}} = \tan\tfrac{1}{2}w + \frac{1}{3}\tan^3\tfrac{1}{2}w$$

where $\tan\frac{1}{2}v = F_2 \tan\frac{1}{2}w$, then F_1 and F_2 must be functions of both the eccentricity and the anomaly, and they must each reduce to unity as the eccentricity approaches unity.

If we let $y = \tan\frac{1}{2}E$, so that $y^2 = u$, then the second equation of ((3,34)) becomes

$$\frac{k(t-T)}{q^{3/2}} = \frac{2}{(1-e)^{3/2}}\left[(1-e)\,y\,\left(1 - \frac{2}{3}y^2 + \frac{3}{5}y^4 - \ldots\right) + \frac{(1+e)}{3}y^3\left(1 - \frac{6}{5}y^2 + \frac{9}{7}y^4 - \ldots\right)\right] \qquad ((3,35))$$

Also
$$E = 2\arctan y = 2y\left(1 - \frac{1}{3}y^2 + \frac{1}{5}y^4 - \ldots\right)$$
$$\sin E = \frac{2y}{1+y^2} = 2y\left(1 - \quad y^2 + \quad y^4 - \ldots\right).$$

Substitute these into the third equation of ((3,34)):

$$\frac{k(t-T)}{q^{3/2}} = \frac{2}{(1-e)^{3/2}}\,y\left[\frac{1-e}{1} - \frac{1-3e}{3}y^2 + \frac{1-5e}{5}y^4 - \ldots\right] \qquad ((3,36))$$

Notice that ((3,36)) agrees with ((3,35)), as indeed it should, and so we have a clue to aid in passing from Kepler's equation to a form having a linear and a cubic term. If E is considered to be a quantity of the first order, then $E - \sin E$ is known to be of the third order. We also notice that

$$\frac{3}{4}(E - \sin E) = y^3\left(1 - \frac{6}{5}y^2 + \frac{9}{7}y^4 - \ldots\right)$$

which is the same as the second line of ((3,35)). Also the first order term is factored by (1 - e). This leads us to recognize that $E - e\sin E$ must be grouped into two parts, such as, for example, $(1-e)\sin E + (E - \sin E)$, so as to have terms of the first and third order which, perhaps, can be transformed to the parabolic form. Thus Kepler's equation becomes

$$\frac{k(t-T)}{q^{3/2}} = \frac{(1-e)\sin E + (E - \sin E)}{(1-e)^{3/2}} = \frac{\sqrt{2}}{B}\left[\left(\frac{2A}{1-e}\right)^{1/2} + \frac{1}{3}\left(\frac{2A}{1-e}\right)^{3/2}\right] \qquad ((3,37))$$

where we see, by comparing coefficients, that this equation will be satisfied if we write

$$\frac{2\sqrt{A}}{B} = \sin E, \qquad \frac{4A^{3/2}}{3B} = (E - \sin E) \qquad \text{or} \qquad B = \frac{2\sqrt{A}}{\sin E}, \qquad A = \frac{3}{2}\,\frac{E - \sin E}{\sin E}.$$

Finally
$$B\,\frac{k(t-T)}{\sqrt{2}\,q^{3/2}} = \tan\tfrac{1}{2}w + \frac{1}{3}\tan^3\tfrac{1}{2}w,$$

where $\tan^2\frac{1}{2}w = \frac{2A}{1-e}$. To find the relationship between v and w, we may write

$$\tan\tfrac{1}{2}v = cC\tan\tfrac{1}{2}w = cC\sqrt{\frac{2A}{1-e}} = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}E,$$

and these equations will be satisfied if $c = \sqrt{\frac{1+e}{2}}$, $C = \frac{\tan\frac{1}{2}E}{\sqrt{A}}$.

Both B and C may be tabulated as functions of A, so that with the aid of such a table we may proceed to solve for $\tan\frac{1}{2}w$ by successive approximations, beginning with $B=1$. The value of $\tan\frac{1}{2}w$ obtained by using $B=1$ allows us to compute $A = \frac{1}{2}(1-e)\tan^2\frac{1}{2}w$, and this yields a value of B which permits a more accurate solution for $\tan\frac{1}{2}w$. This is repeated until A reaches its final value; then $\tan\frac{1}{2}v = cC\tan\frac{1}{2}w$.

Now A is a quantity of the second order, $\frac{A}{1-e}$ is of the order of $\tan^2\frac{1}{2}v$, and

$$B^2 = \frac{4A}{\sin^2E} = 6\frac{E-\sin E}{\sin^3E} = \frac{1-E^2/20+\ldots}{1-E^2/2+\ldots} = 1 + \frac{9}{20}E^2 + \ldots.$$

If the two functions whose ratio is B could be brought to have equal second order terms, then B would differ from unity by only a quantity of the fourth order; and the solution for $\tan\frac{1}{2}w$ would converge much more rapidly, for the value of B would be much less sensitive to the errors in the successive approximations to A. The problem is thus reduced to an attempt to eliminate the 9 which we have found in the numerator of the last term. This fact was first brought to the writer's attention by A.D. Maxwell.

Gauss must have perceived, with the perspicacity that marked his genius, that the denominator of B depends upon the original grouping in Kepler's equation. There is not much latitude in the arrangement of $E - e\sin E$, because the portion factoring $(1-e)$ must be of the first order and the remaining portion must be of the third order. If we notice that the third order term in ((3,35)) is factored by $(1+e)$, we might then try $E - e\sin E = \frac{1}{2}(1-e)(E+\sin E) + \frac{1}{2}(1+e)(E-\sin E)$. After we carry through the same development as above, we obtain $B = 1 + E^2/5 + \ldots$, which is better than we had before, and suggests the course to pursue in making further trials.

In a manner which is not indicated, Gauss arrived at the following arrangement of Kepler's equation, (notice that at this point we discard the previous definitions of A, B, C and c):

$$k(t-T)\left(\frac{1-e}{q}\right)^{3/2} = E - e\sin E = (1-e)\frac{9E+\sin E}{10} + \frac{(1+9e)}{10}(E-\sin E)$$
$$= \frac{\sqrt{2}}{B}\left[(1-e)(2A)^{1/2} + \frac{(1+9e)}{30}(2A)^{3/2}\right]$$

where $\frac{2\sqrt{A}}{B} = \frac{9E+\sin E}{10}$, $A = 15\frac{(E-\sin E)}{9E+\sin E}$.

Finally
$$aB\frac{k(t-T)}{\sqrt{2}\,q^{3/2}} = \tan\tfrac{1}{2}w + \tfrac{1}{3}\tan^3\tfrac{1}{2}w, \tag{(3,38)}$$

where $\tan^2\frac{1}{2}w = \frac{1+9e}{5(1-e)}A$, $a = \sqrt{\frac{1+9e}{10}}$. Also

$$\tan\tfrac{1}{2}v = cC\tan\tfrac{1}{2}w = cC\sqrt{\frac{1+9e}{5(1-e)}A} = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}E$$

so that $c = \sqrt{\frac{5(1+e)}{1+9e}}$, $C = \frac{\tan\frac{1}{2}E}{\sqrt{A}}$, and if we write $b = \frac{5(1-e)}{1+9e}$, then $A = b\tan^2\frac{1}{2}w$.

Now $A = (\frac{1}{2}E)^2 + \ldots$, $B = 1 + \frac{3}{2800}E^4 + \ldots$, so that even when the eccentric anomaly is as large as 60°, the error in the first approximation is only about one part in a thousand. Beyond this value of E, Kepler's equation may be solved in the usual way, since $\frac{dE}{dM}$ is less than two.

We notice that
$$r = \frac{p}{1+e\cos v} = \frac{q(1+e)}{1+e\frac{1-x^2}{1+x^2}} = \frac{q(1+x^2)}{1+y^2}.$$

If we write $D = \dfrac{1}{1+\tan^2\frac{1}{2}E} = \frac{1}{2}(1+\cos E)$, this may also be tabulated as a function of A, along with B and C. Then

$$r = qD(1+\tan^2\tfrac{1}{2}v), \quad r\cos v = qD(1-\tan^2\tfrac{1}{2}v), \quad r\sin v = 2qD\tan\tfrac{1}{2}v. \qquad ((3{,}39))$$

As the eccentricity increases from an elliptic to a hyperbolic value, u becomes negative and where the odd powers of u all had negative signs in the formulas of an elliptic orbit, these terms now become positive and u is written $\frac{e-1}{e+1}x^2$. The quantity $\tan\frac{1}{2}v = x$ is a real quantity, therefore y becomes imaginary and so does E. The equation for hyperbolic motion may be written as

$$k(t-T)\left(\frac{e-1}{q}\right)^{3/2} = (e-1)\frac{9F+\sinh F}{10} + \frac{1+9e}{10}(\sinh F - F)$$

and a development may be obtained similar to the elliptic case, with (e - 1), sinh F, and - F corresponding to (1 - e), - sin E, and E, respectively. Let

$$A = 15\frac{\sinh F - F}{9F+\sinh F}, \quad B = \frac{20\sqrt{A}}{9F+\sinh F}, \quad C = \frac{\tanh\frac{1}{2}F}{\sqrt{A}}, \quad D = \tfrac{1}{2}(1+\cosh F),$$

and all the results are of the same form as above.

Tables of the functions B, C and D with the argument A for both the ellipse and the hyperbola have been computed by the author and are given in the appendix. Logarithmic tables were also given by Marth, Astronomische Nachrichten, vol. 43, p. 115. In the following formulas, read the upper sign for an ellipse and the lower sign for a hyperbola.

$$a = \sqrt{\frac{1+9e}{10}}, \quad b = \pm\frac{5(1-e)}{1+9e}, \quad c = \sqrt{\frac{5(1+e)}{1+9e}}, \quad u = \pm\frac{1-e}{1+e}\tan^2\tfrac{1}{2}v$$

$$A = b\tan^2\tfrac{1}{2}w = u \mp 0.8u^2 + 0.686u^3 \mp 0.6u^4 + \ldots, \quad \tan\tfrac{1}{2}v = cC\tan\tfrac{1}{2}w$$

$$r = qD(1+\tan^2\tfrac{1}{2}v), \quad r\cos v = qD(1-\tan^2\tfrac{1}{2}v), \quad r\sin v = 2qD\tan\tfrac{1}{2}v \qquad ((3{,}40))$$

$$B\frac{ak(t-T)}{\sqrt{2}\,q^{3/2}} = \tan\tfrac{1}{2}w + \tfrac{1}{3}\tan^3\tfrac{1}{2}w$$

Given t, to find $\tan\frac{1}{2}v$: Begin with B = 1 (or whatever better estimated value is known), solve for $\tan\frac{1}{2}w$ and A; then B is given by the table. Repeat the solution until A reaches its final value; then take C from the table, and compute $\tan\frac{1}{2}v$.

Given $\tan\frac{1}{2}v$, to find T: Begin with u and the value of A given by the series in u. Take C from the table, compute $\tan\frac{1}{2}w = \frac{\tan\frac{1}{2}v}{cC}$, and $A = b\tan^2\frac{1}{2}w$. Repeat until A reaches its final value; then take B from the table, and solve for (t - T) from the last equation of ((3,40)).

We have now surveyed the general characteristics of the motion of an object about the Sun, we have described one set of six parameters or elements which serve to define the orbit, and we have developed methods for locating the object in its orbit. We shall now investigate properties of the apparent path of the object upon the sky as seen from the Earth, and the relations between the observations and the heliocentric motion. As described in Chapter 2, the situation as viewed from the Earth is the same as if we were dealing with the motion of a point constrained to move upon the surface of a unit sphere. The space relations which exist are defined by the equations ((2,6)). Since our problem has six independent unknowns and each observation is able to yield only two measured data, we see that we shall need at least three observations to determine all the elements. This is a necessary condition, based upon very elementary considerations, but as yet we have ascertained nothing about sufficient conditions.

From the triangle formed by the Sun, the object, and the Earth, we may express the conditions contained in ((2,6)) by the vector equation

$$\mathbf{r} + \mathbf{R} = \boldsymbol{\rho} = \rho\boldsymbol{\rho}^* \qquad ((3{,}41))$$

where $\boldsymbol{\rho}^*$ is a unit vector directed outward along the line of sight. (Read $\boldsymbol{\rho}$ as rho.)

Also
$$\frac{d\mathbf{p}^*}{dt} = \frac{d\mathbf{p}^*}{ds}\frac{ds}{dt} = V\mathbf{T}, \tag{(3,42)}$$

where $\mathbf{T}$ is the unit vector tangent to the apparent path on the unit sphere and V is the linear speed along the apparent path.

Let us consider the path upon the surface of the unit sphere as a space curve, and let $\mathbf{N}$ be a unit vector normal to $\mathbf{p}^*$ and $\mathbf{T}$ such that $\mathbf{N}\cdot(\mathbf{p}^*\times\mathbf{T}) = +1$. Then $-\mathbf{p}^*$ is directed along the principal normal of the space curve and $\mathbf{N}$ is directed along the binormal. Referred to this moving frame of reference,

$$\frac{d\mathbf{T}}{ds} = -\mathbf{p}^* + K\mathbf{N} \tag{(3,43)}$$

where K is the geodesic curvature. At any instant, $\mathbf{T}$ defines a great circle on the unit sphere, and K is a measure of the rate of motion of this great circle. Whenever $K = 0$, this great circle remains instantaneously fixed and the apparent motion of the object upon the celestial sphere is along this great circle.

Differentiate ((3,42)), and substitute from ((3,43)):

$$\frac{d^2\mathbf{p}^*}{dt^2} = \frac{dV}{dt}\mathbf{T} + V^2\frac{d\mathbf{T}}{ds} = \frac{dV}{dt}\mathbf{T} + KV^2\mathbf{N} - V^2\mathbf{p}^* \tag{(3,44)}$$

Now differentiate ((3,41)) twice and in the left hand member substitute for the acceleration according to the law of gravitation as applied to the Earth and the object separately.

$$-\frac{\mathbf{R}}{R^3} - \frac{\mathbf{r}}{r^3} = -\mathbf{R}\left(\frac{1}{R^3} - \frac{1}{r^3}\right) - \frac{\mathbf{p}}{r^3} = \frac{d^2\mathbf{p}}{dt^2} = \rho\frac{d^2\mathbf{p}^*}{dt^2} + 2\frac{d\rho}{dt}\frac{d\mathbf{p}^*}{dt} + \frac{d^2\rho}{dt^2}\mathbf{p}^*. \tag{(3,45)}$$

Operate upon both sides of this equation by $\cdot(\mathbf{p}^*\times\mathbf{T})$ and substitute from ((3,42)) and ((3,44)).

$$\left(\frac{1}{r^3} - \frac{1}{R^3}\right)\mathbf{R}\cdot(\mathbf{p}^*\times\mathbf{T}) = \rho KV^2\mathbf{N}\cdot(\mathbf{p}^*\times\mathbf{T})$$

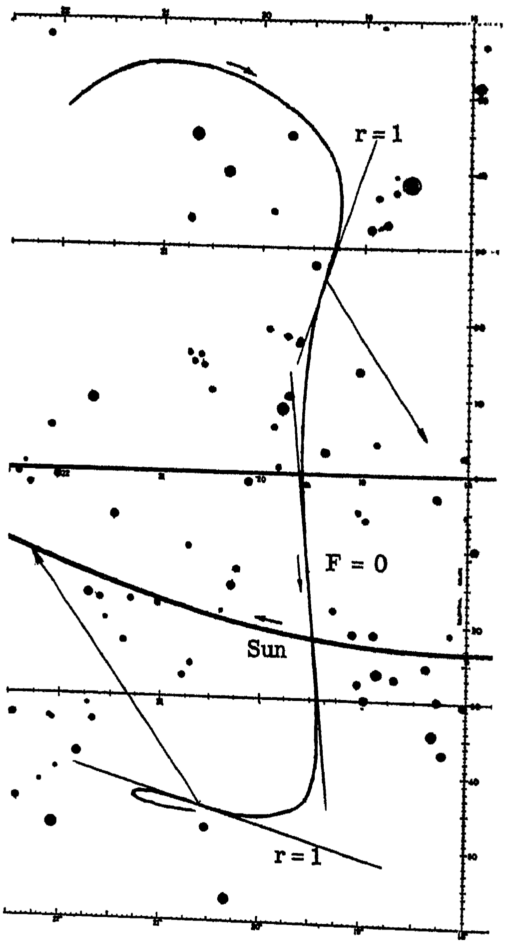
or

$$\left(\frac{1}{r^3} - \frac{1}{R^3}\right)R\sin F = KV^2\rho \tag{(3,46)}$$

where F is the angular distance from the Sun to the great circle which is tangent to the apparent path.

This equation ((3,46)) is the basis of what is known as Lambert's theorem on the curvature of the apparent path. In the first place, we notice that by combining this equation, which contains the dynamical conditions and is of the form $\rho = A + B/r^3$, with the square of the first equation ((3,41)), which contains the geometrical conditions and is of the form $r^2 = \rho^2 - 2(\mathbf{R}\cdot\mathbf{p}^*)\rho + R^2$, we have two equations in the two unknowns r and ρ, the solution of which will give us the distance along the line of sight to the newly discovered object. All the other quantities in the equations are known, either from the observations or the solar coordinates; but this is not the most practical method of solution.

According to our definitions, if K and F are both positive, or both negative, this means that the Sun lies on the concave side of the apparent path of the object, and equation ((3,46)) requires that R be greater than r. Conversely, if K and F are of opposite signs, the Sun is on the convex side of the path and r is then greater than R. This leads to the general statement of Lambert's theorem: when the apparent path is convex toward the Sun, then r is greater than R; and when the apparent



path is concave toward the Sun, then r is less than R. All the possibilities of this theorem are illustrated in the figure on the previous page showing a plot of the path of Comet Cunningham, 1940 - c.

The limiting case between these two situations may arise from either one of two causes. First, we may have $r = R$. This happens, of course, twice whenever a comet comes to perihelion at a distance of less than one astronomical unit. In the right hand member of ((3,46)), we need not consider the trivial case, $\rho = 0$. It is very unlikely that $V = 0$, so that $K = 0$ is the only remaining possibility. This is what is naturally to be expected at the inflection points which the path must have if the Sun is to pass from the convex to the concave side or *vice versa*.

Secondly, we may have $F = 0$. This exists continually for objects which have no inclination to the ecliptic. But it may also happen fortuitously for any object when the Sun happens to cross the great circle determined by **T**. In the figure, this happened on about January 12th, just as the comet was about to cross the ecliptic. Irrespective of the source of the zero on the left hand side of the equation, it is evident that we are, in such cases, unable to solve for ρ because its coefficient has vanished. If we know that $F \neq 0$, we may assume that $r = R$ and thus obtain the solution notwithstanding. But if $F = 0$ and the observations lie on a great circle, it is not possible to make a general solution for the orbit from three such observations. This is naturally to be expected, for the three observations are no longer independent, and so they provide only five independent data instead of the necessary six. To get a general solution in such cases, it is necessary to use methods which depend upon four observations.

The student must recognize that in practical numerical computation there is no sharp line of demarcation between a quantity which is exactly zero and one which is extremely small. It will be found in this problem that for short arcs the middle position is seldom very distant from the great circle joining the two outer positions, and therefore the coefficient of ρ is usually a small number, sometimes very small. This causes an unavoidable degree of uncertainty in the solution, perhaps so great that the result is of little value. It is always necessary to be alert to the situations in which the results are poorly determined. In general, such situations may arise from an improper mathematical formulation of the problem, such as, for example, the determination of a small angle from its cosine instead of its sine; or they may be due to the inherent physical nature of the problem. In the latter case, there is no way to increase the determinacy of the solution. In the present problem it is necessary, in order to overcome such situations, to obtain more observations over a longer arc and a longer interval of time.

From the converse point of view, it is apparent that when three observations are relatively close together and also subject to some variation on account of the uncertainty due to the unavoidable errors and limitations of the observations, then there must be many sets of numerical values of the six elements, each of which will give practically as reliable a representation of these same observations as any other set. The equations indicate this situation by showing us that they have no strong preference for the particular set of values we obtain.

In summary, we may be sure that no matter what situation confronts us in practice, it is certain to be exhibited in the factors of ((3,46)), if only we interpret them correctly. The prospect of a satisfactory solution from three observations depends primarily upon their deviation from great circle motion. In case $K=0$ but $F \neq 0$, then we may assume $r=R$ and obtain a solution; but if $F = 0$, then a solution from three observations is impossible.

CHAPTER 4

THE METHOD OF LA PLACE

Μὴ αἱρεῖσθε λεπτὸν παράδειγμα· τὸ ῥῶ οὐδενὶ ἰσοῦται.

The motion of a body in the solar system has been examined analytically, and we have discovered the general nature of the result to be expected in any particular case. We shall now consider the actual numerical problem of determining a preliminary orbit of a newly discovered body from three observations. The method which follows most readily from the elementary processes of calculus, with which the student is familiar, was proposed by that great celestial mechanician of the 18th century, La Place. It is essentially a Taylor's series expansion in which we need to find the appropriate numerical coefficients. This, in effect, is a solution of the fundamental differential equation by means of the process which is usually referred to in textbooks on differential equations as the solution by series.

This method which La Place devised for the determination of a preliminary orbit makes no direct use of the knowledge of the solution which we have obtained from our previous developments. The plan of attack is very general and, indeed, as we shall see later, lends itself very well even to the determination of a preliminary disturbed solution in cases where that may become necessary. The curvatures of the space path which the object follows must be reflected in some way in the curvatures of its apparent path on the sky. If we were to set down the geometrical and differential relationships which must exist between the two, and, in addition, impose the conditions of the law of gravitation in the differential equations, we may then attempt to infer from the observable path the nature of the actual path of the body around the Sun.

It is not difficult to realize that the expressions which would represent each observed coordinate as a literal function of the six unknown elements, so as to provide six equations in the six unknowns, would be so complicated as to be wholly unmanageable. Instead of attempting to manipulate the analytical solution which we have derived for the Two Body Problem, we shall proceed entirely de nova with a different method of attack, but this time fortified with the foreknowledge we have already gained concerning the result.

The point of view may be described in still another way. We have a differential equation to solve which is expressed in heliocentric coordinates, but the boundary conditions or the arbitrary constants of integration can be determined only after a transformation of variables to geocentric, observed coordinates. The differential equation which we have to solve is (when the quantities are expressed in astronomical units, solar masses, and 1/k mean solar days) simply

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} \tag{(4,1)}$$

Write the solution in the form of a Taylor's series:

$$\mathbf{r} = \mathbf{r}_0 + \tau\,\mathbf{r}_0' + \frac{\tau^2}{2!}\mathbf{r}_0'' + \frac{\tau^3}{3!}\mathbf{r}_0''' + \frac{\tau^4}{4!}\mathbf{r}_0^{iv} + \dots \tag{(4,2)}$$

Now our differential equation ((4,1)) will permit us to eliminate $\mathbf{r}_0''$ in terms of $\mathbf{r}_0$ and r_0. Similarly, the higher derivatives of ((4,2)) may be eliminated by successive differentiations of the differential equation ((4,1)) and substitutions. Let us first define

$$\frac{\mathbf{r}_0\cdot\mathbf{r}_0}{r_0^5} = \frac{1}{r_0^3} = \mu \qquad \frac{\mathbf{r}_0\cdot\mathbf{r}_0'}{r_0^2} = \frac{r_0'}{r_0} = \sigma \qquad \frac{\mathbf{r}_0'\cdot\mathbf{r}_0'}{r_0^2} = \omega$$

Then (omitting the zero subscripts) we have

$$\frac{d\mu}{dt} = \mu' = \frac{2\,\mathbf{r}\cdot\mathbf{r}'}{r^5} - \frac{5\,\mathbf{r}\cdot\mathbf{r}}{r^6}\,r' = -\,3\mu\sigma$$

$$\frac{d\sigma}{dt} = \sigma' = \frac{\mathbf{r}'\cdot\mathbf{r}'}{r^2} + \frac{\mathbf{r}\cdot\mathbf{r}''}{r^2} - 2\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}\,r' = \omega - \mu - 2\sigma^2$$

$$\frac{d\omega}{dt} = \omega' = \frac{2\mathbf{r}'\cdot\mathbf{r}''}{r^2} - \frac{2\mathbf{r}'\cdot\mathbf{r}'}{r^3}\,r' = -\,2\sigma(\omega + \mu)$$

$$\begin{aligned}
\frac{1}{2!}\mathbf{r}^{\mathrm{ii}} &= -\frac{1}{2}\mu\mathbf{r}\\
\frac{1}{3!}\mathbf{r}^{\mathrm{iii}} &= -\frac{1}{6}(\mu'\mathbf{r} + \mu\mathbf{r}')\\
&= +\frac{1}{2}\mu\sigma\mathbf{r} - \frac{1}{6}\mu\mathbf{r}'\\
\frac{1}{4!}\mathbf{r}^{\mathrm{iv}} &= +\frac{1}{8}(\mu'\sigma\mathbf{r} + \mu\sigma'\mathbf{r} + \mu\sigma\mathbf{r}') - \frac{1}{24}(\mu'\mathbf{r}' + \mu\mathbf{r}'')\\
&= +\frac{1}{24}\mu(3\omega - 2\mu - 15\sigma^2)\mathbf{r} + \frac{1}{4}\mu\sigma\mathbf{r}'\\
\frac{1}{5!}\mathbf{r}^{\mathrm{v}} &= -\frac{1}{8}\mu\sigma(3\omega - 2\mu - 7\sigma^2)\mathbf{r} + \frac{1}{120}\mu(9\omega - 8\mu - 45\sigma^2)\mathbf{r}'\\
\frac{1}{6!}\mathbf{r}^{\mathrm{vi}} &= \frac{\mu}{720}[(630\omega - 420\mu - 945\sigma^2)\sigma^2 - (22\mu^2 - 66\mu\omega + 45\omega^2)]\,\mathbf{r}\\
&\quad - \frac{1}{24}\mu\sigma(6\omega - 5\mu - 14\sigma^2)\mathbf{r}'
\end{aligned} \qquad ((4,3))$$

Substitute all the expressions of ((4,3)) into ((4,2)), and write

$$\mathbf{r} = f\,\mathbf{r}_0 + g\,\mathbf{r}_0' \qquad ((4,4))$$

where $f = 1 - \frac{1}{2}\mu\tau^2 + \frac{1}{2}\mu\sigma\tau^3 + \ldots$, and $g = \tau - \frac{1}{6}\mu\tau^3 + \ldots$.
After $\mathbf{r}^{\mathrm{vi}}$ the formulas become so complicated as to be impractical.

We have now shown that if we are able to find the position vector $\mathbf{r}_0$ and the velocity vector $\mathbf{r}_0'$ at the time t_0, we shall be able to find $\mathbf{r}$ at any other time t, provided only that the f and g series converge. In other words, we have obtained an expansion of the function $\mathbf{r}(t)$ about the point t_0, and $\mathbf{r}_0$ and $\mathbf{r}_0'$ are the constants of integration of our differential equation ((4,1)). As with $\mathbf{h}$ and $\mathbf{e}$, these two vectors have six components which correspond to six scalar constants of integration. These may be considered to be another set of elements which will also define the orbit. It remains only to find $\mathbf{r}_0$ and $\mathbf{r}_0'$ from the observations.

All methods which are similar to the method of La Place are based upon the following principles. The geometrical conditions are contained in the equation

$$\mathbf{r} = \rho\mathbf{p}^* - \mathbf{R}. \qquad ((4,5))$$

Differentiate this equation twice with respect to t, and impose the dynamical conditions by substituting for the accelerations in accordance with the law of gravitation from ((4,1)). Thus

$$\mathbf{r}' = \rho'\mathbf{p}^* + \rho\mathbf{p}^{*\prime} - \mathbf{R}' \qquad ((4,6))$$

$$\mathbf{r}'' = \rho''\mathbf{p}^* + 2\rho'\mathbf{p}^{*\prime} + \rho\mathbf{p}^{*\prime\prime} - \mathbf{R}'' = -\mu\mathbf{r} = \mu(\mathbf{R} - \rho\mathbf{p}^*) \qquad ((4,7))$$

Multiply both sides of ((4,7)) by $(\mathbf{p}^* \times \mathbf{p}^{*\prime})\cdot$

$$\rho\,[\mathbf{p}^* \times \mathbf{p}^{*\prime}\cdot\mathbf{p}^{*\prime\prime}] = [\mathbf{p}^* \times \mathbf{p}^{*\prime}\cdot\mathbf{R}''] + [\mathbf{p}^* \times \mathbf{p}^{*\prime}\cdot\mathbf{R}]/r^3 \qquad ((4,8))$$

By squaring ((4,5)), we also obtain

$$r^2 = \rho^2 + R^2 - 2(\mathbf{p}^*\cdot\mathbf{R})\rho \qquad ((4,9))$$

The only unknowns in these two equations ((4,8)) and ((4,9)) are ρ and r, and the main object of the computation is to find the values which exist at t_0, usually the time of the middle observation. Each of the triple scalar products in ((4,8)) may be evaluated from the observational data and the

known solar coordinates. It is of little consequence, in practice, whether we determine $\mathbf{R}''$ from the solar coordinates by numerical differentiation or substitute $-\mathbf{R}/R^3$. If we write

$$\mathbf{p}^* = \mathbf{p}_0^* + \tau\mathbf{p}_0^{*\prime} + \tfrac{1}{2}\tau^2\mathbf{p}_0^{*\prime\prime} + \ldots, \qquad ((4,10))$$

then each observation gives a pair of values of $\mathbf{p}^*$ and τ, so that we may solve for the unknowns $\mathbf{p}_0^*$, $\mathbf{p}_0^{*\prime}$ and $\mathbf{p}_0^{*\prime\prime}$ on the right hand side.

It should be recognized that while three unknowns on the right hand side may be determined from three observations, the resulting values are only approximate since there are more terms which should be taken into account in the series ((4,10)), even though they are not needed in the equation ((4,8)) with which we are concerned. If more than three observations are available at the time the solution is made, then more equations may be written down and the higher order terms of $\mathbf{p}_0^*$ eliminated first, thus giving more accurate values for the ones which are needed.

When more than the necessary minimum number of three observations is available, there is afforded the opportunity to scrutinize the observations with the view to testing their consistency and possibly detecting errors due to faulty reduction or transmission. This is best accomplished by examining the higher order derivatives which are obtained numerically from the observations. Divide the difference in the coordinates by the difference in the times for all the consecutive pairs of observations (except when they are very close together); this gives the mean rate of change at the mean time. Then treat these values in the same manner in order to obtain 2nd derivatives, etc. Each successive set of values should be smooth, but the smoothness will be destroyed if any of the observations are appreciably in error. An example of this practice has been referred to on page 23 and may be found in the Astronomical Journal, vol. 45, p. 127.

In actual numerical applications we would be obliged to solve ((4,10)) for the values of the components of these vector quantities upon each of the coordinate axes separately. The components of $\mathbf{p}^*$ are the direction cosines of the observations. The components of $\mathbf{p}_0^{*\prime}$ and $\mathbf{p}_0^{*\prime\prime}$ then enable us to evaluate the triple scalar products in ((4,8)). After ρ_0 is known, we find $\mathbf{r}_0$ from ((4,5)) and $\mathbf{r}_0'$ from ((4,6)). To find ρ_0', we multiply ((4,7)) by $\cdot(\mathbf{p}^* \times \mathbf{p}^{*\prime})$; then we have

$$-2\rho'[\mathbf{p}^*\times\mathbf{p}^{*\prime}\cdot\mathbf{p}^{*\prime\prime}] = [\mathbf{p}^*\times\mathbf{p}^{*\prime\prime}\cdot\mathbf{R}''] + [\mathbf{p}^*\times\mathbf{p}^{*\prime\prime}\cdot\mathbf{R}]/r^3.$$

The solution by means of the above formulas is not a wholly impractical scheme, in fact, it is the method given by Moulton, Celestial Mechanics, Chapter VI. There the student will find all the formulas written in terms of third order determinants, corresponding to our triple scalar products. Also the above solution to the problem is no different, in principle, from the method devised by Harzer, Astronomische Nachrichten, vol. 141, p. 177, or its modification as promulgated by Leuschner, Publications of the Lick Observatory, vol. 7. This method is based upon the curtate distance, $\sigma = \rho\cos\delta$, as the principal unknown instead of ρ. This is equivalent to using cylindrical coordinates and suffers from the disadvantage of having a singularity or a pole at the celestial poles of the sky. The equations ((4,6)) to ((4,9)) are essentially expressed in direction components and they have no such disadvantage anywhere in the sky.

But more ingenious than any of the other methods which depend upon the above principles is the one given in 1931 by Stumpff, Astronomische Nachrichten, vol. 243, p. 317, and vol. 244, p. 433. This method derives its principal advantage from the use of the ratios of the direction cosines and the resultant reduction of all the determinants from the third to the second order. Let

$$U = \frac{y+Y}{x+X} = \tan\alpha, \qquad V = \frac{z+Z}{x+X} = \sec\alpha\tan\delta, \qquad P = Y - UX, \qquad Q = Z - VX. \qquad ((4,11))$$

Cross-multiply the equation for U and V, introduce P and Q, and differentiate twice:

$$\begin{aligned} y &= Ux - P & z &= Vx - Q \\ y' &= U'x + Ux' - P' & z' &= V'x + Vx' - Q' \\ y'' &= U''x + 2U'x' + Ux'' - P'' & z &= V''x + 2V'x' + Vx'' - Q'' \end{aligned} \qquad ((4,12))$$

Substitute the dynamical conditions for each coordinate, or each component of ((4,1)), into the two bottom equations of ((4,12)):

$$\tfrac{1}{2}U''x + U'x' = \tfrac{1}{2}P'' + P/2r^3$$

$$\tfrac{1}{2}V''x + V'x' = \tfrac{1}{2}Q'' + Q/2r^3$$

Let $$D = \tfrac{1}{2}U''V' - \tfrac{1}{2}V''U'.$$

Then
$$\begin{aligned} D\,x &= (\tfrac{1}{2}P''V' - \tfrac{1}{2}Q''U') + (PV' - QU')/2r^3. \\ D\,x' &= (\tfrac{1}{2}Q''\tfrac{1}{2}U'' - \tfrac{1}{2}P''\tfrac{1}{2}V'') + (\tfrac{1}{2}U''Q - \tfrac{1}{2}V''P)/2r^3. \\ r^2 &= x^2 + y^2 + z^2 \\ &= (1 + U^2 + V^2)x^2 - 2(UP + VQ)x + (P^2 + Q^2). \end{aligned} \qquad ((4,13))$$

We now have enough equations to solve ((4,13)) for all the unknowns. Approximate numerical values of the coefficients at t_o, the time of the middle observation, may be obtained from the observations by means of the same principle as was employed in ((4,10)). Write the Taylor's series for the first and third observations in the form

$$\begin{aligned} (W_1 - W_o)/\tau_1 &= W_o' + \tfrac{1}{2}W_o''\tau_1 = (W,1) \\ (W_3 - W_o)/\tau_3 &= W_o' + \tfrac{1}{2}W_o''\tau_3 = (W,3) \end{aligned} \quad \text{then} \quad \begin{aligned} W_o'(\tau_3 - \tau_1) &= \tau_3(W,1) - \tau_1(W,3) \\ \tfrac{1}{2}W_o''(\tau_3 - \tau_1) &= (W,3) - (W,1) \end{aligned} \qquad ((4,14))$$

where W denotes U, V, P or Q, and $\tau_1 = k(t_1 - t_o)$.

This simple demonstration represents a complete solution to the problem of determining a preliminary orbit and is, in itself, the collection of formulas. The first and last equations of ((4,13)) may be solved by iteration, beginning with some approximate value of r^2. This value is substituted in the right hand side of the first equation, thus giving a value for x; the use of Table X of Planetary Coordinates will be an aid in this step. This value of x is substituted into the right hand side of the third equation, thus giving a better value for r^2, and then the process is repeated until the unknowns reach their final values.

It must be noted that the definitions of P and Q insure that these equations are also satisfied by the motion of the Earth, so that the computer must be careful to guard against deriving this fictitious solution instead of the real one. It will be helpful to plot the two curves on a set of x- and r^2-axes. On such axes the r^2 equation is a parabola, and the other equation is asymptotic horizontally to the x-axis and vertically at the value of its own constant term. This curve is more readily plotted by using r^2 as the independent variable. One intersection must correspond to the negative of the solar coordinates, a second will be recognized as giving a spurious result, and the other will give good initial values with which to begin the solution by iteration. Furthermore, the slopes of the two curves in the neighborhood of this intersection will enable the computer to visualize the convergence of the iteration process and perhaps improve it by jumping to better values, or the reason for its divergence in case it should fail. After r_o^2 is known, x_o' may be found from the second equation of ((4,13)), and then y_o, y_o', z_o and z_o' from ((4,12)).

As stated before, the principle advantage of this method lies in the fact that most of the quantities such as D or the coefficients in ((4,13)) have formulas which are of such a form that the entire numerical value may be accumulated in the product dials or the quotient dials of the computing machine. The controlling factor in the whole solution is the value of D. This corresponds to the coefficient of ρ in the left hand member of ((4,8)). If this is extremely small, it indicates either that the time interval $(t_1 - t_o)$ is too small to make the solution very determinate or that K is nearly zero in Lambert's equation ((3,46)) and a satisfactory solution cannot be obtained. In the latter case, a method of overcoming the difficulty by using four observations has been presented by the author in the Astronomical Journal, vol. 48, p. 122.

We shall now simulate the situation in which a newly discovered minor planet is reported and only three observations on a relatively short arc are announced. Let these be the first three of the five which have already been partly reduced on page 24. We wish to determine as much information as possible concerning the elements and future motion of the object in order that it might be identified and that its positions may be predicted with reasonable accuracy so as to aid in making further observations.

The computations follow; the zero subscript, corresponding to the time of the middle observation, has been omitted:

i	1	2	3
	-0.9217386	-0.9460249	-0.9667071
$\mathbf{R}_1$	+0.3782763	+0.3214131	+0.2612860
	+0.1640270	+0.1393582	+0.1132835

	τ_1		τ_3	τ_3	$-\tau_1$
	-0.0671933		+0.0692971	+0.0692971	+0.0671933
	W_1	W_2	W_3	(W,1)	(W,3)
U	-0.2395896	-0.2507026	-0.2625363	-0.165389	-0.170768
V	-0.0663341	-0.0813223	-0.0971067	-0.223061	-0.227779
P	+0.1574373	+0.0842422	+0.0074903	-1.089321	-1.107577
Q	+0.1028843	+0.0624253	+0.0194098	-0.602128	-0.620740
S	1.0304384	1.0341495	1.0384389	$\tau_3 - \tau_1$ = +0.1364904	

We have written $S = \sec\alpha \sec\delta$, then $\rho = S(x + X)$.

$\frac{1}{2}U''$	$\frac{1}{2}P''$	P'	P	U'
-0.039409	-0.133753	-1.098308	+0.0842422	-0.168037
$\frac{1}{2}V''$	$\frac{1}{2}Q''$	Q'	Q	V'
-0.034567	-0.136361	-0.611291	+0.0624253	-0.225384

$$D = +0.0030736$$
$$x = +2.3530 - 1.3823/r^3$$
$$x' = +0.2441 + 0.0735/r^3$$
$$r^2 = +1.0694651x^2 + 0.0523926x + 0.0109937$$

x	r^2	$1/r^3$	x	r^2	$1/r^3$	x	Δ		
2.3530	6.055	0.06712	2.353	6.0555	0.067108	2.2602	-928	+873	
2.2602	5.593	0.07560	2.253	5.5576	0.076325	2.2475	- 55		-25
2.2485	5.536	0.07677	2.153	5.0812	0.087307	2.2323	+793	+848	((A))
2.2469	5.528	0.07694	2.2466						
2.2466	5.5265	0.07697							

$$0 = -55 + (860.5 - 12.5n)n$$

r	r'				
+2.2466000	+0.249757	r^2	5.5265156	ω	+0.0909693
-0.6474707	+0.658181	r	2.3508542	σ	+0.0206651
-0.2451240	+0.084632	μ	0.0769703	σ^2	+0.0004270

n	$f^{(n)}$	$g^{(n)}$	τ_1^n	τ_3^n
0	+1.0	0.0	+1.0	+1.0
1	0.0	+1.0	-0.0671933	+0.0692971
2	-0.0384851	0.0	+0.0045149	+0.0048021
3	+0.0007953	-0.0128284	-0.0003034	+0.0003328
4	+0.0003610	+0.0003977	+0.0000204	+0.0000231
5	-0.0000231	+0.0001179	-0.0000014	+0.0000016
		f	+0.9998260	+0.9998155
		g	-0.0671894	+0.0692928
		x + X	+1.3076895	+1.2967848
		y + Y	-0.3133045	-0.3404580
		z + Z	-0.0867407	-0.1259309
		ρ	1.3474923	1.3466332
		tan α	-0.2395863	-0.2625401
		sin δ	-0.0643719	-0.0935154
		α	$23^h 06^m 06^s.40$	$23^h 01^m 09^s.49$
		δ	$-3^\circ 41' 26''.8$	$-5^\circ 21' 57''.2$
		(O - C)	$-0^s.04$ $-0''.6$	$+0^s.05$ $+0''.7$

Beginning with x = +2.3530 at ((A)), we have shown at the left the numerical values resulting from the successive substitutions in the process of iteration. In cases where the convergence is too slow, the modification shown at the right will usually be found much more effective. After we find that x = +2.3530 yields a new value which is +2.2602 or a correction of -928, we compute the corrections corresponding to the equidistant values +2.2530 and +2.1530. Then these corrections

are inversely interpolated by Stirling's formula ((1,16)) to the value of n corresponding to a zero correction. Finally, we obtain the components of the position vector and the velocity vector at the epoch, and by means of the f and g series ((4,3)) we are able to compare our solution with the observations. We have designated by $f^{(n)}$ and $g^{(n)}$ the coefficients of τ^n in the f and g series.

The number of decimal places and significant figures to be used in each case is dependent largely upon the particular circumstances. The observations are usually given to an accuracy of about 0.0000005 radians, so that 7 decimals in the trigonometric functions and solar coordinates are more than sufficient. The time intervals in the present case are of the order of 0.1 and the coefficients are divided by the small value of D, so that even the 4 decimal values which we have given are meaningless in the last place. However, once a value of x_0 is adopted, the values of all the other quantities must be kept consistent with it and they must be carried to the full accuracy that is needed for the comparison with the observations, usually 6 or 7 decimal places.

It is easy to see that the method which we have used for our preliminary solution would break down if the observations were in the neighborhood of 6^h or 18^h right ascension, due to the large or even meaningless values obtained for the derivatives of U. Stumpff has recommended to overcome this difficulty by rotating the coordinate system about the z-axis through an angle α_0, the right ascension of the middle observation. This is not a certain remedy, because the observations may be in the neighborhood of the celestial pole and so the difficulty would shift to the V's. It would be possible, of course, to rotate the coordinate system arbitrarily so that the x-axis is directed toward the middle observation, but all these remedies have the disadvantage that after the solution is obtained, the results must be rotated back again. These rotations are accomplished by means of operations with third order matrices, and in addition to the increased opportunities for committing numerical errors, the total amount of work is greater than if we had used the direction cosines and third order determinants in the first place, since that method has no such singularities at any point in the sky.

We shall find a way to overcome this difficulty very simply if we examine its source. We have taken the ratios of the direction cosines, and the difficulty arises whenever the one we have placed in the denominator becomes very small. Let us divide the whole sky into three regions in such a way that in each region we may choose the ratios so that they are each less than unity in absolute magnitude. Thus we have three cases, as follows:

Case	I	II	III
Independent variable	x	y	z
U	$\tan\alpha$	$\cot\alpha$	$\cos\alpha\cot\delta$
V	$\sec\alpha\tan\delta$	$\csc\alpha\tan\delta$	$\sin\alpha\cot\delta$
P	$Y - UX$	$X - UY$	$X - UZ$
Q	$Z - VX$	$Z - VY$	$Y - VZ$
S	$\sec\alpha\sec\delta$	$\csc\alpha\sec$	$\csc\delta$
Dependent variables	$y = Ux - P$ $z = Vx - Q$	$x = Uy - P$ $z = Vy - Q$	$x = Uz - P$ $y = Vz - Q$

The necessary changes in all the formulas are readily perceived. In any application, choose the case for which the values of U_0 and V_0 are each less than unity. Our illustrative example above comes under Case I.

The solution which we have obtained will automatically satisfy the middle observation, due to the way in which the dependent variables are determined. The way in which the first and third observations are computed from the solution by means of the f and g insures that the dynamical conditions are satisfied. But these observations are not necessarily represented exactly, due to the errors in the derivatives which were determined numerically from the observational data and used in the solution. This means that we do not necessarily have the best result that can be obtained from the observations, and we still have a problem with four degrees of variability.

There are also other factors which contribute to the discrepancy between our preliminary solution and the true orbit, so that under any circumstances we shall eventually have to improve

our results after more observations become available. It should be recognized that each observation is subject to some error in the last place, due to the physical limitations of seeing and measurement, the scale of the micrometer or photographic plate, the appearance of the image, and the errors in the positions of the comparison stars, aside from any avoidable errors of carelessness or accident. The observed position may therefore be considered merely as the center of a minute circle on the sky whose radius is at least so large as to give a good statistical probability that it encloses the point which is the true position and which we would prefer to use if it were exactly known. The true orbit therefore passes through three points, one lying at some unknown place within each of these minute circles, whereas in the computation we have used their centers. When the three circles are relatively close together, this permits the computed orbit to deviate considerably from the true orbit. In other words, when D has several zeros to the right of the decimal point and therefore a fewer number of significant figures, the results can not be expected to be accurate to any more significant figures.

The most that the computer can do, in any event, is to determine elements which will give a satisfactory representation of the observations, even though he recognizes that the results may be highly uncertain. The process by which residuals are generally reduced will be presented in Chapter 6, but we may consider now a method of solution which is applicable to short arcs. This method depends upon a principle which does not entail residuals in the observations. We shall write our equations in such a way that all the geometrical conditions, i.e. the observations, are exactly satisfied and the purpose of the solution is to find the values of the unknowns which will also satisfy the dynamical conditions. In other words, we shall use rigorous equations which include both geometrical and dynamical conditions, and which are solved by repeated substitutions.

In the equation $y_1 = U_1 x_1 - P_1$, substitute

$$y_1 = U_1(f_1 x_0 + g_1 x_0') - P_1 = f_1(U_0 x_0 - P_0) + g_1 y_0',$$

or

$$\begin{aligned} g_1 y_0' &= f_1(U_1 - U_0)x_0 + U_1 g_1 x_0' + f_1 P_0 - P_1 \\ g_3 y_0' &= f_3(U_3 - U_0)x_0 + U_3 g_3 x_0' + f_3 P_0 - P_3 \\ g_1 z_0' &= f_1(V_1 - V_0)x_0 + V_1 g_1 x_0' + f_1 Q_0 - Q_1 \\ g_3 z_0' &= f_3(V_3 - V_0)x_0 + V_3 g_3 x_0' + f_3 Q_0 - Q_3. \end{aligned} \qquad ((4,15))$$

The last three equations are obtained in the same manner as the first one. These equations apply in the event of Case I. In either of the other two cases, the correct equations are obtained by the appropriate interchange of x, y and z. By subtraction, we obtain $(g_3 - g_1)y_0'$ and $(g_3 - g_1)z_0'$; by eliminating the left hand members in pairs, we obtain two equations in the remaining unknowns, x_0 and x_0'. The f's and g's are assumed to be known from whatever preliminary solution is available. In order for the method to operate successfully it is necessary that they are not seriously affected by the errors in the coordinates and velocities. Therefore we may write out the four equations with the numerical values of the coefficients, including the f's and g's, and solve for x_0 and x_0', then y_0' and z_0', and finally y_0 and z_0. These quantities are then used to recompute the f's and g's, and the whole process is repeated until it converges to the final values. In the successive recomputations of the f's and g's, it is necessary to correct each of the times of observation for planetary aberration: $t\,(\text{true}) = t\,(\text{obs.}) - 0^{d}\!.00577\,S\,(x + X)$.

It is apparent that no preliminary solution is required if one is willing to begin with $r = 2.5$ or 3.0 and $r' = 0$ (assuming the object is a minor planet). This will give rather crude values for the f's and g's and the iteration process will take longer to converge. The inexperienced computer will be well advised not to follow a procedure such as this by rule of thumb. It requires some insight to judge at each stage of the work whether the iteration process is approaching the solution or whether the computer is "going around in circles". The method suffers from another disadvantage which would not become apparent to the reader until later: due to the four degrees of variability (as contrasted with the two which we shall have in the Gaussian method) the equations leave the unknowns for which we are solving more poorly determined in most cases.

This method is to be grouped with those in which the geometrical conditions are always satisfied (through the P's and Q's of all three observations) but the dynamical conditions are not satisfied until the last iterative cycle shows that the f's and g's which were used in the previous

step prove also to be the correct values for the final step. It seems scarcely necessary to remark that in the event a solution from three observations is intrinsically impossible, this will become evident from the vanishing of the determinant of the coefficients in trying to solve the equations for x_0 and x'_0, or in the continuing oscillation or divergence of the successive solutions, without any convergence to a final limit. This method has only a limited application in practice and is not to be highly recommended, but it does illustrate one example of the principles described in the Introduction.

Returning to our example on page 44, we see that our residuals are fairly satisfactory. They are of about the order of magnitude of the errors of the individual observations, and so we may consider the first part of our task completed. If the residuals were too large, we would be faced with the further problem of first reducing them by some process of improvement before proceeding with the rest of the computation. We still have two remaining parts to our problem: the determination of the elements in their usual form as described on page 31, and the computation of an ephemeris. We shall therefore next derive a number of useful relationships which enable us to transform from the position and velocity vectors to the usual orbital elements.

In this problem the given data are (omitting the zero subscripts):

$$\mathbf{r}\cdot\mathbf{r} = r^2 = x^2 + y^2 + z^2, \quad \mathbf{r}\cdot\mathbf{r}' = (r\,r') = xx' + yy' + zz', \quad \mathbf{r}'\cdot\mathbf{r}' = G^2 = x'^2 + y'^2 + z'^2,$$
$$\mathbf{r}\times\mathbf{r}' = (yz' - zy')\mathbf{i} + (zx' - xz')\mathbf{j} + (xy' - yx')\mathbf{k}, \tag{(4,16)}$$

where we have already seen that the last expression is a vector which is normal to the plane of the orbit and whose length is $\sqrt{p}$.

The following expressions have been encountered in the previous chapter; we wish to transform them so as to derive the elliptic elements from our given data.

$$G^2 = 2/r - 1/a, \qquad r = a(1 - e\cos E).$$

From the derivative of Kepler's equation: $(1 - e\cos E)\,dE = a^{-3/2}dt$ or $E' = 1/r\sqrt{a}$.

From the derivative of r: $(r\,r') = r\,a\,e\sin E\,E' = \sqrt{a}\,e\sin E$.

Thus: $$\frac{r}{a} = 2 - rG^2 = 1 - e\cos E \quad \text{and} \quad e\sin E = \frac{(r\,r')}{\sqrt{a}}. \tag{(4,17)}$$

The computations proceed from ((4,16)) in the following order:

$$r^2,\ r,\ G^2,\ 1 - e\cos E,\ a,\ \sqrt{a},\ P = a^{3/2},\ e\cos E,\ e\sin E,\ e^2,\ e,\ \tan E,\ E,\ M,\ n.$$

If E is expressed in degrees, then: $M = E - (e\sin E)\,57\overset{\circ}{.}2957795$ and $n = 0\overset{\circ}{.}9856106/P$,

where the numerical value of k corresponds to an augmented mass of the Sun.

If the eccentricity is so large that Kepler's equation is to be avoided, and the semi-major axis a is extremely large and poorly determined, then we may write

$$G^2 = \frac{2}{r} - \frac{(1 - e^2)}{p}, \quad r = \frac{p}{1 + e\cos v}, \quad r' = \frac{p\,e\sin v\,v'}{(1 + e\cos v)^2} = \frac{r^2 e\sin v\,v'}{p} = \frac{e\sin v}{\sqrt{p}}$$

Then $(2 - rG^2)p = r(1 - e^2)$, and e^2 is eliminated by substituting $(e\cos v)^2 + (e\sin v)^2$. Thus

$$e\cos v = \frac{p - r}{r}, \qquad e\sin v = \frac{(r\,r')\sqrt{p}}{r}$$
$$r(2 - rG^2)p = r^2 - (p^2 - 2pr + r^2) - (r\,r')^2 p$$

or
$$p = r^2G^2 - (r\,r')^2. \tag{(4,18)}$$

The computations now proceed from ((4,16)) in the following order:

$$r^2,\ r,\ (r\,r'),\ G^2,\ p,\ \sqrt{p},\ e\cos v,\ e\sin v,\ e^2,\ e,\ \tan\tfrac{1}{2}v,\ T.$$

If it should become necessary, either in the course of the computations or in some theoretical development, to transform from E to v or from v to E, the following transformation equations may be employed. (Other transformations, including those involving Fourier series expansions and Bessel functions, may be found in various treatises on celestial mechanics.)

$$r\cos v = a(\cos E - e) \qquad r\sin v = a\sqrt{1-e^2}\sin E \qquad r = \frac{a(1-e^2)}{1+e\cos v} = a(1-e\cos E)$$

$$\cos v = \frac{\cos E - e}{1 - e\cos E} \qquad \sin v = \frac{\sqrt{1-e^2}\sin E}{1-e\cos E} \qquad \cos v + e = \frac{(1-e^2)\cos E}{1-e\cos E} \tag{(4,19)}$$

$$\cos E = \frac{\cos v + e}{1+e\cos v} \qquad \sin E = \frac{\sqrt{1-e^2}\sin v}{1+e\cos v} \qquad \cos E - e = \frac{(1-e^2)\cos v}{1+e\cos v}$$

We have now obtained the elliptic elements, a, e, M, or p, e, T; it remains to determine the position of the ellipse in space. It is possible, however, to determine a position in the ellipse at some other time, t, without direct reference to the remaining elements. In accordance with the usual notation, we shall define **P**, **Q**, **R** to be three mutually perpendicular unit vectors referred to the equatorial coordinate system and directed toward the perihelion, toward $v = +90°$ of true anomaly in the orbit plane, and along the normal, respectively. The reader will need to be careful not to confuse this dual use of the notation **R**: when it represents the unit vector normal to the orbit plane it has the components R_x, R_y, R_z; when it represents the geocentric position vector of the Sun its components are the solar coordinates, X, Y, Z. Taken in conjunction with the context, there will be no ambiguity. Also let $\mathbf{A} = a\mathbf{P}$ and $\mathbf{B} = b\mathbf{Q}$. Then

$$\mathbf{r} = \mathbf{P}(r\cos v) + \mathbf{Q}(r\sin v) = \mathbf{A}(\cos E - e) + \mathbf{B}\sin E$$
$$\mathbf{r}' = \frac{\mathbf{Q}(\cos v + e) - \mathbf{P}\sin v}{\sqrt{p}} = \frac{\mathbf{B}\cos E - \mathbf{A}\sin E}{r\sqrt{a}} \tag{(4,20)}$$

Since $\mathbf{r} = f\mathbf{r}_0 + g\mathbf{r}_0'$, we have:

$$\mathbf{r}\times\mathbf{r}_0' = f\,\mathbf{r}_0\times\mathbf{r}_0' \quad \text{and} \quad \mathbf{r}_0\times\mathbf{r} = g\,\mathbf{r}_0\times\mathbf{r}_0'. \tag{(4,21)}$$

Also

$$\mathbf{r}_0\times\mathbf{r}_0' = \mathbf{A}\times\mathbf{B}\left[\frac{(\cos E_0 - e)\cos E_0 + \sin E_0\sin E_0}{r_0\sqrt{a}}\right] = \frac{\mathbf{A}\times\mathbf{B}}{a^{3/2}}$$
$$\mathbf{r}\times\mathbf{r}_0' = \mathbf{A}\times\mathbf{B}\left[\frac{(\cos E - e)\cos E_0 + \sin E\sin E_0}{r_0\sqrt{a}}\right]$$
$$\mathbf{r}_0\times\mathbf{r} = \mathbf{A}\times\mathbf{B}\,[(\cos E_0 - e)\sin E - (\cos E - e)\sin E_0]$$

Substitute these expressions and equate the coefficients of $\mathbf{A}\times\mathbf{B}$ in ((4,21)).

$$f = \frac{-e\cos E_0 + \cos E\cos E_0 + \sin E\sin E_0}{1 - e\cos E_0} = \frac{e\cos v + \cos v\cos v_0 + \sin v\sin v_0}{1+e\cos v}$$
$$g = [(\cos E_0 - e)\sin E - (\cos E - e)\sin E_0]\,a^{3/2} = (\sin v\cos v_0 - \cos v\sin v)\frac{r_0 r}{\sqrt{p}} \tag{(4,22)}$$

The right hand expressions may be obtained either by a similar process or by direct substitution.

We may also solve for two other functions, f' and g', which will enable us to find $\mathbf{r}'$ at any time, t, from the formula

$$\mathbf{r}' = f'\mathbf{r}_0 + g'\mathbf{r}_0'$$

In this case we obtain

$$\mathbf{r}'\times\mathbf{r}_0' = f'\mathbf{r}_0\times\mathbf{r}_0' \quad \text{and} \quad \mathbf{r}_0\times\mathbf{r}' = g'\mathbf{r}_0\times\mathbf{r}_0' \tag{(4,23)}$$

and by means of ((4,20)) and ((4,23)):

$$\mathbf{r}'\times\mathbf{r}_0' = \mathbf{A}\times\mathbf{B}\left[\frac{\sin E_0\cos E - \sin E\cos E_0}{r\,r_0\,a}\right]$$
$$\mathbf{r}_0\times\mathbf{r}' = \mathbf{A}\times\mathbf{B}\left[\frac{(\cos E_0 - e)\cos E + \sin E_0\sin E}{r\sqrt{a}}\right]$$

$$f' = \frac{\sin E_0\cos E - \sin E\cos E_0}{(1-e\cos E_0)(1-e\cos E)\,a^{3/2}} = \frac{\sin v_0(\cos v + e) - \sin v(\cos v_0 + e)}{p^{3/2}}$$
$$g' = \frac{-e\cos E + \cos E\cos E_0 + \sin E\sin E_0}{1-e\cos E} = \frac{e\cos v_0 + \cos v\cos v_0 + \sin v\sin v_0}{1+e\cos v_0} \tag{(4,24)}$$

The series expressions for f' and g' are obtained by differentiating the f and g series with respect

to τ_1. $\quad f' = 2\tau_1 f^{(2)} + 3\tau_1^2 f^{(3)} + \ldots \quad$ and $\quad g' = 1 + 3\tau_1^2 g^{(3)} + 4\tau_1^3 g^{(4)} + \ldots . \qquad$ ((4,25))

In case we have many observations to represent, there would be a considerable simplification in the formulas for f and g if t_0 were T, the time of perihelion passage. Then

$$\cos E_0 = 1.0, \quad \sin E_0 = 0.0, \quad f = \frac{\cos E - e}{1 - e}, \quad g = a^{3/2}(1 - e)\sin E,$$

$$\mathbf{r} = \frac{\mathbf{r}_0}{1 - e}(\cos E - e) + \mathbf{r}'_0 a^{3/2}(1 - e)\sin E = \mathbf{A}(\cos E - e) + \mathbf{B}\sin E, \qquad ((4,26))$$

where the subscript zero now refers to $t_0 = T$.

In order to transform from the given position and velocity vectors, $\mathbf{r}_0$ and $\mathbf{r}'_0$, at some given time t_0 to the desired values at $t = T$, i.e. from ((4,16)) to ((4,26)), we make use of ((4,22)) and ((4,24)). Since $t = T$, $\cos E = 1.0$ and $\sin E = 0.0$. Then

$$\mathbf{r}(T) = f\,\mathbf{r}_0 + g\,\mathbf{r}'_0 = \frac{(1 - e)\cos E_0}{1 - e\cos E_0}\mathbf{r}_0 - (1 - e)\sin E_0\, a^{3/2}\mathbf{r}'_0$$

$$\mathbf{r}'(T) = f'\mathbf{r}_0 + g'\mathbf{r}'_0 = \frac{\sin E_0}{(1 - e)(1 - e\cos E_0)\,a^{3/2}}\mathbf{r}_0 + \frac{\cos E_0 - e}{(1 - e)}\mathbf{r}'_0$$

or

$$\frac{\mathbf{r}(T)}{(1 - e)} = \mathbf{A} = \frac{\cos E_0}{1 - e\cos E_0}\mathbf{r}_0 - a^{3/2}\sin E_0\,\mathbf{r}'_0$$

$$a^{3/2}(1 - e)\,\mathbf{r}'(T) = \mathbf{B} = \frac{\sin E_0}{1 - e\cos E_0}\mathbf{r}_0 + a^{3/2}(\cos E_0 - e)\,\mathbf{r}'_0 \qquad ((4,27))$$

which enables us to obtain $\mathbf{A}$ and $\mathbf{B}$ from any $\mathbf{r}_0$ and $\mathbf{r}'_0$.

The corresponding equations for $\mathbf{P}$ and $\mathbf{Q}$ are:

$$\mathbf{P} = \frac{\cos v_0 + e}{p}\mathbf{r}_0 - \frac{r_0 \sin v_0}{\sqrt{p}}\mathbf{r}'_0$$

$$\mathbf{Q} = \frac{\sin v_0}{p}\mathbf{r}_0 + \frac{r_0 \cos v_0}{\sqrt{p}}\mathbf{r}'_0 \qquad ((4,28))$$

The components of the unit vectors, $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, referred to the equatorial coordinate system are known as the vectorial constants for the equator. They are discussed in greater detail by Smiley in the Astronomical Journal, vol. 40, page 31. If $\mathbf{A}$, $\mathbf{B}$, and T are given, they constitute another complete set of elements. The size, shape, and orientation of the orbit in space are defined by $\mathbf{A}$ and $\mathbf{B}$, and T permits the determination of the position of the object in the orbit. These correspond to seven scalar quantities, but only six are independent, for we must always satisfy the condition that $\mathbf{A}\cdot\mathbf{B} = 0$.

We have already seen that $\mathbf{r} \times \mathbf{r}' = \sqrt{p}\,\mathbf{R}$, so that if we compute the components of this equation according to ((4,16)) we shall have not only the direction of the normal, but also an independent check, since the sum of the squares of the components must equal p. We may let $\mathbf{V} = \mathbf{r}\times\mathbf{R}$; then $\mathbf{V}$ is a vector whose absolute magnitude is r and it lies in the orbit plane 90° of true anomaly behind $\mathbf{r}$. Then

$$r\mathbf{P} = \mathbf{r}\cos v + \mathbf{V}\sin v \quad \text{and} \quad r\mathbf{Q} = \mathbf{r}\sin v - \mathbf{V}\cos v. \qquad ((4,29))$$

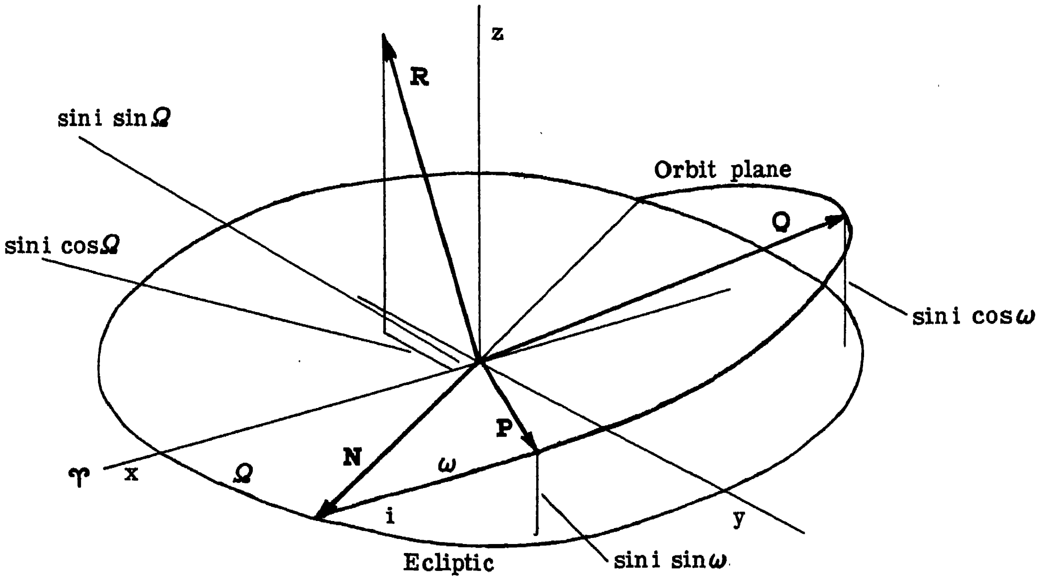
In actual computation we would compute the three components of each of these equations, e.g.

$$P_x = \frac{x\cos v + V_x \sin v}{r}, \text{ etc.}$$

The elements, i, Ω, and ω, are referred to the ecliptic, therefore if we rotate the coordinate system about the x-axis through an angle ϵ, we shall have

$$\begin{Bmatrix} R_x \\ R_y \\ R_z \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{Bmatrix} = \begin{Bmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{Bmatrix} \qquad ((4,30))$$

which are the components of $\mathbf{R}$ in the ecliptic coordinate system. Let $\mathbf{N}$ be a unit vector directed toward the ascending node; then it has the components ($\cos\Omega$, $\sin\Omega$, 0) in the ecliptic coordinate system, and therefore



$$\begin{aligned} \mathbf{N}\cdot\mathbf{P} &= \cos\omega = \cos\Omega\, P_x + \sin\Omega(\cos\epsilon\, P_y + \sin\epsilon\, P_z) \\ \mathbf{N}\cdot\mathbf{Q} &= -\sin\omega = \cos\Omega\, Q_x + \sin\Omega(\cos\epsilon\, Q_y + \sin\epsilon\, Q_z) \end{aligned} \qquad ((4,31))$$

where the ( )'s are the y-components of $\mathbf{P}$ and $\mathbf{Q}$ in the ecliptic coordinate system.

Alternatively,

$$\begin{aligned} \sin i \sin\omega &= \cos\epsilon\, P_z - \sin\epsilon\, P_y \\ \sin i \cos\omega &= \cos\epsilon\, Q_z - \sin\epsilon\, Q_y \\ \sin i \sin\Omega &= [P_y(\sin i\cos\omega) - Q_y(\sin i \sin\omega)]\sec\epsilon \\ \sin i\cos\Omega &= [P_x(\sin i\cos\omega) - Q_x(\sin i\sin\omega)] \end{aligned} \qquad ((4,32))$$

and we have an independent check: $P_x(\sin i \sin\omega) + Q_x(\sin i \cos\omega) + \cos i(\sin i \sin\Omega) = 0$. These equations will be seen to be true from the adjoining figure. The first two of ((4,32)) come from the z-component of $\mathbf{P}$ and $\mathbf{Q}$ in the ecliptic coordinate system. The other two are obtained by taking the x- and y-components of $\mathbf{R} = \mathbf{P}\times\mathbf{Q}$.

If we wish to obtain i, Ω, and ω from $\mathbf{r}_0$ and $\mathbf{r}_0'$ without passing through the intermediary of the vectorial constants, we may transform the rectangular coordinates and velocities directly to the ecliptic coordinate system by means of the operation

$$\begin{Bmatrix} x_0 & x_0' \\ y_0 & y_0' \\ z_0 & z_0' \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{Bmatrix} = \begin{Bmatrix} \bar{x} & \bar{x}' \\ \bar{y} & \bar{y}' \\ \bar{z} & \bar{z}' \end{Bmatrix}$$

Then

$$\begin{aligned} \sin i \sin\Omega &= (\bar{y}\bar{z}' - \bar{z}\bar{y}')/\sqrt{p} & \qquad r\sin u &= \pm|\mathbf{N}\times\mathbf{r}| = \bar{z}\csc i \\ -\sin i\cos\Omega &= (\bar{z}\bar{x}' - \bar{x}\bar{z}')/\sqrt{p} & \qquad r\cos u &= \mathbf{N}\cdot\mathbf{r} \\ \cos i &= (\bar{x}\bar{y}' - \bar{y}\bar{x}')/\sqrt{p} & \qquad \omega &= u - v \end{aligned}$$

The converse problem of finding $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ from the given elements, i, Ω, and ω, may be solved by the following formulas and arrangement of the computation:

$$\begin{aligned} p &= \sin\omega\,(\cos i\cos\Omega) + \cos\omega\sin\Omega \\ q &= \cos\omega\,(\cos i \cos\Omega) - \sin\omega\sin\Omega \\ P_x &= -[\sin\omega\,(\cos i\sin\Omega) - \cos\omega\cos\Omega] \\ Q_x &= -[\cos\omega(\cos i\sin\Omega) + \sin\omega\cos\Omega] \\ P_y &= \cos\epsilon\; p - \sin\epsilon\,(\sin i \sin\omega) \\ Q_y &= -[\sin\epsilon\,(\sin i\cos\omega) - \cos\epsilon\; q] \\ P_z &= \cos\epsilon\,(\sin i\sin\omega) + \sin\epsilon\; p \\ Q_z &= \sin\epsilon\; q + \cos\epsilon\,(\sin i\cos\omega) \end{aligned}$$

$\sin i$	$\cos i$	
$\cos i \cos\Omega$	$\sin\omega$	$\cos\Omega$
$\sin\Omega$	$\cos\omega$	$\cos i \sin\Omega$
p	$\sin\epsilon$	q
$\sin i \sin\omega$	$\cos\epsilon$	$\sin i\cos\omega$
P_x	Q_x	
P_y	Q_y	
P_z	Q_z	

These formulas may also be derived with the aid of the figure. The auxiliary quantities, p and q, are the y-components of $\mathbf{P}$ and $\mathbf{Q}$ in the ecliptic coordinate system. The formulas are arranged so that the first factor of any product should be set on the keyboard and each quantity is obtained either as the value of a second order determinant by cross-multiplication of the form $a_{11}a_{22} - a_{12}a_{21}$ or as the sum of two products when the factors appear beside each other, as $a_{11}a_{21} + a_{12}a_{22}$. Finally $\mathbf{R} = \mathbf{P}\times\mathbf{Q}$, and as checks: $\mathbf{P}\cdot\mathbf{Q} = 0$, $\mathbf{P}^2 = \mathbf{Q}^2 = 1$, $R_x = \sin i \sin\Omega$, $R_z \cos\epsilon - R_y \sin\epsilon = \cos i$.

An ephemeris may be computed in a number of different ways. We shall not yet attempt to consider all these methods, since some depend upon results which will be developed subsequently. For the present, we may simply use ((4,20)) in conjunction with Kepler's equation. The standard ephemeris dates which are to be used have been designated by the International Astronomical Union as being the midnight following the Julian Day number which is evenly divisible by the number of days in the interval. Continuing from our preliminary solution on page 44, we shall now determine the vectorial constants, the classical elements, and a sample of the ephemeris.

					A		**B**
r_0	2.3508542	G^2	0.5027434		+2.1524591		+1.8587144
a	2.8734707	P	4.8709107		-1.8557600		+2.1095943
$\sqrt{a}$	1.6951315	n	$0^\circ.2023463$		-0.4241561		+0.2025339
$e \sin E$	+0.0673728	e^2	0.0376181	$\sin i \sin\omega$	+0.1212379	$\sin\epsilon$	+0.39788118
$e \cos E$	+0.1818764	$\cos\phi$	0.9810107	$\sin i \cos\omega$	-0.2318474	$\cos\epsilon$	+0.91743695
$1 - e\cos E$	0.8181236	e	0.1939539	$\sin i \sin\Omega$	+0.0640667	$\sec\epsilon$	+1.0899932
		e°	$11^\circ.11274$	$\sin i \cos\Omega$	-0.2538112		
$\tan E$	+0.3704318	$\cos E$	+0.9377300	$\sin i$	+0.261772	i	$15^\circ.1752$
E	$+20^\circ.32623$	$(\cos E - e)$	+0.7437761	$\tan\Omega$	-0.252419	Ω	165.8334
M	+16.46605	$\sin E$	+0.3473650	$\tan\omega$	-0.524215	ω	152.3358
$\mathbf{A} = +1.1461960\, \mathbf{r}_0 - 1.6919839\, \mathbf{r}_0'$				a	2.8734707	M_0	$+16^\circ.46605$
$\mathbf{B} = +0.4245874\, \mathbf{r}_0 + 3.6228670\, \mathbf{r}_0'$				e	0.1939539	t_0	Sept. $2^d.8989$

All the elements are collected at the lower right for easy reference.

	Sept. 11	Sept. 19	Sept. 27	Oct. 5
M°	+18.10528	+19.72405	+21.34282	+22.96159
E°	+22.32691	+24.29648	+26.25951	+28.21559
$\cos E$	+0.9250314	+0.9114285	+0.8967993	+0.8811748
$(\cos E - e)$	+0.7310775	+0.7174746	+0.7028454	+0.6872209
$\sin E$	+0.3798906	+0.4114583	+0.4424376	+0.4727905
$x + X$	+1.29677	+1.30791	+1.33439	+1.37643
$y + Y$	-0.35590	-0.38833	-0.42158	-0.45335
$z + Z$	-0.14667	-0.18841	-0.23047	-0.27182
ρ	1.35270	1.37729	1.41825	1.47444
$\tan\alpha$	-0.27445	-0.29691	-0.31593	-0.32936
$\sin\delta$	-0.10843	-0.13680	-0.16250	-0.18435

1935 UT	α (1950.0)		δ	
Sept. 11.0	$22^h\ 58^m.6$		$-6^\circ\ 13'$	
		4.7		99
Sept. 19.0	22 53.9		-7 52	
		4.0		89
Sept. 27.0	22 49.9		-9 21	
		2.8		76
Oct. 5.0	22 47.1		-10 37	

This is all the accuracy that is needed for a finding ephemeris, and the computations could have been carried to one less decimal place.

CHAPTER 5

THE METHODS OF GAUSS AND OLBERS

Νήπιοι, πρὸς ταῦτα μὴ διατείνεσθε.

The method which Gauss devised for the determination of a preliminary orbit reflects both his thorough insight into the essentials of the problem and his genious for reducing these to practical numerical processes. He was led to develop his solution to this problem by the discovery of the first minor planet and the necessity of predicting its position after its conjunction with the Sun. Prior to this time there had been little need for a general preliminary solution, since the newly discovered objects had all been comets. These could generally be represented by parabolic orbits, with the consequent simplification of the problem to five unknowns, since e = 1. Such cases were adequately provided for by Olbers' method. When Uranus was discovered, it was not recognized as a planet until after the failure of attempts to represent its motion by a parabola.

It is possible to simplify the problem still further by assuming that the object moves in a circular orbit. In this case there is no eccentricity nor longitude of perihelion, so that the problem is reduced to the determination of only four unknowns, say i, Ω, a, and the longitude in the orbit at some epoch. Two observations are therefore all that are needed to provide the necessary data, and the solution may be determined from a single equation with one unknown. We have

$$|\mathbf{r}_i \times \mathbf{r}_j| = r_i r_j \sin(v_j - v_i) = a^2 \sin(M_j - M_i) = \sqrt{a}\,\tau\left[1 - \frac{1}{6}\frac{\tau^2}{a^3} + \frac{1}{120}\frac{\tau^4}{a^6} - \ldots\right] \qquad ((5,1))$$

since $M_j - M_i = \tau/a^{3/2}$. Let ρ_i be the independent variable to be determined from this equation. With some assumed value of ρ_i, we have

$$\rho_i^2 - 2(\mathbf{p}_i^* \cdot \mathbf{R}_i)\rho_i + R_i^2 = a^2 = \rho_j^2 - 2(\mathbf{p}_j^* \cdot \mathbf{R}_j)\rho_j + R_j^2 \qquad ((5,2))$$

Find a, solve the quadratic for ρ_j, write down the components of $\mathbf{r}_i = \mathbf{p}_i^*\rho_i - \mathbf{R}_i$ and $\mathbf{r}_j = \mathbf{p}_j^*\rho_j - \mathbf{R}_j$ form the components of $\mathbf{r}_i \times \mathbf{r}_j$, and finally evaluate:

$$\Delta(\rho_i) = |\mathbf{r}_i \times \mathbf{r}_j| - \sqrt{a}\,\tau\left[1 - \frac{1}{6}\frac{\tau^2}{a^3} + \frac{1}{120}\frac{\tau^4}{a^6} - \ldots\right] \qquad ((5,3))$$

Compute by trials until $\Delta(\rho_i) = 0$. This value of ρ_i is the solution.

The problem of determining the conditions under which equation ((5,3)) will yield a real solution is a complicated one. It has been examined by Tisserand, and he finds that difficulties may be expected if the time interval is very short, if the observations are too close to opposition, or if the motion exceeds certain limits. For example, if the student attempts to derive a circular orbit from the first and third observations on page 24, he will find this impossible. This is due mainly to the fact that these observations lie near the opposition. If however, the fourth and fifth are used, the following results will be obtained.

$$\mathbf{p}_i^* \begin{matrix} +0.9436933 \\ -0.2928372 \\ -0.1539138 \end{matrix} \qquad \mathbf{p}_j^* \begin{matrix} +0.9209644 \\ -0.3188352 \\ -0.2239835 \end{matrix} \qquad \begin{matrix} r_i^2 = \rho_i^2 + 1.8924673\,\rho_i + 1.0064954 = a^2 \\ r_j^2 = \rho_j^2 + 1.2697772\,\rho_j + 0.9905102 = a^2 \end{matrix}$$

		$\mathbf{r}_i$	$\mathbf{r}_j$			
ρ_i	1.0	+1.9469345	+1.9723540	0.9	+1.8525652	+1.8734703
ρ_j	1.1848740	-0.2914147	+0.0467315	1.0775043	-0.2621310	+0.0809647
a^2	3.8989627	-0.1532526	-0.0812307	3.5197160	-0.1378612	-0.0571817
a	1.9745791	τ^n	$\mathbf{r}_i \times \mathbf{r}_j$	1.8760906		
$\sqrt{a}$	1.4051972	0.4813038	+0.0308335	1.3697046	0.4813038	+0.0261510
$1/6a^3$	0.0216484	0.2316533	-0.1441175	0.0252399	0.2316533	-0.1523460
$1/120a^6$	0.0001406	0.0536633	+0.6657561	0.0001911	0.0536633	+0.6410870
Δ	+.0089336	-0.6729401	+0.6818737	+.0040624	-0.6553963	+0.6594587

0.8	+1.7581958	+1.7737093	0.7	+1.6638265	+1.6728806
0.9691820	-0.2328473	+0.1155017	0.8597003	-0.2035635	+0.1504083
3.1604692	-0.1224698	-0.0329193	2.8212225	-0.1070785	-0.0083972
1.7777709			1.6796495		
1.3333308	0.4813038	+0.0218106	1.2960129	0.4813038	+0.0178149
0.0296634	0.2316533	-0.1593472	0.0351717	0.2316533	-0.1651581
0.0002640	0.0536633	+0.6160780	0.0003711	0.0536633	+0.5907907
-.0006110	-0.6373365	+0.6367255	-.0050057	-0.6187061	+0.6137004

0.7	-50057		
		+43947	
0.8	- 6110		+2787
		+46734	
0.9	+40624		+1978
		+48712	
1.0	+89336		

$$0 = -6110 + 46734n \begin{matrix} +2787\,E_0 \\ +1978\,E_1 \end{matrix} \qquad n = +0.1338, \quad \rho_1 = 0.81338.$$

This value of ρ_1 should now be used to repeat the solution, and if an appreciable residual still exists, the derivative may be interpolated from the above table, and the final correction to ρ_1 is then easily obtained by Newton's method of approximation. It will be observed that this value is not near the solution obtained for ρ on page 44. But neither is the orbit of this object nearly a circle. The solution we have just obtained is therefore fictitious, and, in general, not too much reliance should be placed on circular orbits which are derived for objects for which no more than two observations exist.

The nature of the general solution of the Two Body Problem was already well known in Gauss' time, namely that the object moves about the Sun in an ellipse, and that its position in the ellipse is determined by Kepler's equation. Some progress had been made toward a general solution by Lambert; we shall consider later the theorem which he developed for motion in an ellipse. It forms a bridge between the method of Olbers and the method of Gauss. Assuming the form which the solution will take, we may write (in effect, as Gauss did)

$$\mathbf{r}_2 = c_1 \mathbf{r}_1 + c_3 \mathbf{r}_3 \tag{(5,4)}$$

which states that the radius vector at any time t_2 is some linear combination of the radius vectors at t_1 and t_3. Then

$$\mathbf{r}_1 \times \mathbf{r}_2 = c_3 \mathbf{r}_1 \times \mathbf{r}_3, \quad \mathbf{r}_2 \times \mathbf{r}_3 = c_1 \mathbf{r}_1 \times \mathbf{r}_3$$

From these equations we obtain

$$c_1 = \frac{\mathbf{r}_2 \times \mathbf{r}_3 \cdot \mathbf{R}}{\mathbf{r}_1 \times \mathbf{r}_3 \cdot \mathbf{R}} = \frac{r_2 r_3 \sin(v_3 - v_2)}{r_1 r_3 \sin(v_3 - v_1)} = \frac{[r_2, r_3]}{[r_1, r_3]} \quad \text{and} \quad c_3 = \frac{[r_1, r_2]}{[r_1, r_3]} \tag{(5,5)}$$

where $[r_i, r_j]$ stands for the area of the triangle formed by $\mathbf{r}_i$ and $\mathbf{r}_j$ as sides, and the c's are known as the "triangle ratios". This is a geometrical property which is true for any linear combination of vectors of this form. If we substitute $\mathbf{r} = \boldsymbol{\rho} - \mathbf{R}$ into ((5,4)), we have

$$c_1 \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 + c_3 \boldsymbol{\rho}_3 = c_1 \mathbf{R}_1 - \mathbf{R}_2 + c_3 \mathbf{R}_3. \tag{(5,6)}$$

This is one of the fundamental equations of the Gaussian method. The components of this equation would furnish three equations in the three unknown geocentric distances, provided the c's were known. The c's must be determined in such a way that as the object moves from $\mathbf{r}_1$ to $\mathbf{r}_2$ to $\mathbf{r}_3$ the conditions of motion under the influence of the Sun's gravitation are satisfied.

Gauss has given a method for obtaining the c's in his Theoria Motus Corporum Coelestium, Book 1, Sec. 3, Para. 88, which has the practical advantage of depending directly upon the quantities which are essential in the solution for the orbit. First, let (r_i, r_j) represent the area of the sector of the ellipse contained between the two radius vectors $\mathbf{r}_i$ and $\mathbf{r}_j$. Then $\bar{y}$, known as the "sector-triangle ratio", is defined as follows:

$$\frac{\text{Area of sector}}{\text{Area of triangle}} = \bar{y}_2 = \frac{(r_1, r_3)}{[r_1, r_3]}, \quad \bar{y}_1 = \frac{(r_2, r_3)}{[r_2, r_3]}, \quad \bar{y}_3 = \frac{(r_1, r_2)}{[r_1, r_2]}. \tag{(5,7)}$$

The reader will notice the complementary arrangement of the subscripts; this is characteristic of the notation employed in the analysis of the Gaussian method, since this method lacks the concept of a zero point about which an expansion is developed, such as there is in the method of La Place.

According to the law of areas, the areas of the sectors are proportional to the time, therefore we may substitute

$$c_1 = \frac{(r_2, r_3)}{\bar{y}_1}\frac{\bar{y}_2}{(r_1, r_3)} = \frac{(t_3 - t_2)\,\bar{y}_2}{(t_3 - t_1)\,\bar{y}_1}, \quad c_3 = \frac{(r_1, r_2)}{\bar{y}_3}\frac{\bar{y}_2}{(r_1, r_3)} = \frac{(t_2 - t_1)\,\bar{y}_2}{(t_3 - t_1)\,\bar{y}_3} \qquad (\!(5,8)\!)$$

and Gauss shifts the burden of the problem onto the sector-triangle ratios.

In the ellipse, let

$$v_j - v_i = 2f, \quad v_j + v_i = 2F, \quad E_j - E_i = 2g, \quad E_j + E_i = 2G, \quad b = a\cos\phi = p/\cos\phi$$

where v is the true anomaly, E is the eccentric anomaly, and $e = \sin\phi$. Also

$$\cos v = \frac{\cos E - e}{1 - e\cos E}; \quad 1 + \cos v = 2\cos^2\tfrac{1}{2}v = \frac{2(1-e)\cos^2\frac{1}{2}E}{1 - e\cos E}; \quad \cos\tfrac{1}{2}v = \sqrt{\frac{a(1-e)}{r}}\cos\tfrac{1}{2}E$$

$$1 - \cos v = 2\sin^2\tfrac{1}{2}v = \frac{2(1+e)\sin^2\frac{1}{2}E}{1 - e\cos E}; \quad \sin\tfrac{1}{2}v = \sqrt{\frac{a(1+e)}{r}}\sin\tfrac{1}{2}E$$

Write

$$(C,i) = \sqrt{r_i}\cos\tfrac{1}{2}v_i = \sqrt{a(1-e)}\cos\tfrac{1}{2}E_i, \qquad (S,i) = \sqrt{r_i}\sin\tfrac{1}{2}v_i = \sqrt{a(1+e)}\sin\tfrac{1}{2}E_i,$$
$$(C,j) = \sqrt{r_j}\cos\tfrac{1}{2}v_j = \sqrt{a(1-e)}\cos\tfrac{1}{2}E_j, \qquad (S,j) = \sqrt{r_j}\sin\tfrac{1}{2}v_j = \sqrt{a(1+e)}\sin\tfrac{1}{2}E_j,$$

Then

$$\begin{aligned}
(S,j)(C,i) - (C,j)(S,i) &= b\sin g = \sqrt{r_i r_j}\sin f \\
(S,j)(C,i) + (C,j)(S,i) &= b\sin G = \sqrt{r_i r_j}\sin F \\
(1+e)(C,j)(C,i) + (1-e)(S,j)(S,i) &= p\cos g = \sqrt{r_i r_j}\,(\cos f + e\cos F) \\
(1+e)(C,j)(C,i) - (1-e)(S,j)(S,i) &= p\cos G = \sqrt{r_i r_j}\,(\cos F + e\cos f) \\
(C,j)(C,i) + (S,j)(S,i) &= a(\cos g - e\cos G) = \sqrt{r_i r_j}\cos f \\
(C,j)(C,i) - (S,j)(S,i) &= a(\cos G - e\cos g) = \sqrt{r_i r_j}\cos F
\end{aligned} \qquad (\!(5,9)\!)$$

Also

$$\begin{aligned}
r_i + r_j &= 2a - ae(\cos E_j + \cos E_i) = 2a - 2ae\cos g\cos G = 2a - 2a\cos g\left(\cos g - \frac{\sqrt{r_i r_j}\cos f}{a}\right) \\
&= 2a\sin^2 g + 2\sqrt{r_i r_j}\cos f\cos g.
\end{aligned} \qquad (\!(5,10)\!)$$

Thus far we have derived relationships which depend only upon the geometrical properties of the ellipse. The dynamical conditions will be imposed if we introduce Kepler's equation:

$$\begin{aligned}
k(t_j - t_i)a^{-3/2} &= 2g - e(\sin E_j - \sin E_i) = 2g - 2e\sin g\cos G \\
&= 2g - 2\sin g\left(\cos g - \frac{\sqrt{r_i r_j}\cos f}{a}\right) = 2g - \sin 2g + 2\frac{\sqrt{r_i r_j}}{a}\sin g\cos f
\end{aligned} \qquad (\!(5,11)\!)$$

If we assume, for the moment, that $\mathbf{r}_i$ and $\mathbf{r}_j$ are given at the time t_i and t_j, respectively, then we have two equations in which everything is known except a and g, (since $\mathbf{r}_i \cdot \mathbf{r}_j = r_i r_j \cos 2f$). It will facilitate the solution of these two equations if we introduce the following quantities:

$$\kappa^2 = 4r_i r_j\cos^2 f = 2r_i r_j(1 + \cos 2f) = 2(r_i r_j + x_i x_j + y_i y_j + z_i z_j)$$

$$1 + 2l = \frac{\sqrt{r_j/r_i} + \sqrt{r_i/r_j}}{2\cos f} = \frac{r_i + r_j}{2\sqrt{r_i r_j}\cos f} = \frac{r_i + r_j}{\kappa}, \qquad m^2 = \frac{\tau^2}{(2\sqrt{r_i r_j}\cos f)^3} = \tau^2\kappa^{-3} \qquad (\!(5,12)\!)$$

$$x = \sin^2\tfrac{1}{2}g$$

Then from ((5,10)):

$$\begin{aligned}
a &= \frac{r_i + r_j - 2\sqrt{r_i r_j}\cos f\cos g}{2\sin^2 g} = \frac{2\sqrt{r_i r_j}\cos f(1 + 2l) - 2\sqrt{r_i r_j}\cos f\cos g}{2\sin^2 g} \\
&= \frac{2\sqrt{r_i r_j}\cos f(1 + \sin^2\frac{1}{2}g)}{\sin^2 g} = \frac{\kappa(1+x)}{\sin^2 g}
\end{aligned} \qquad (\!(5,13)\!)$$

Substitute this into ((5,11)) to eliminate a:

$$k(t_j - t_i) = (2g - \sin 2g)a^{3/2} + 2\sqrt{r_i r_j}\cos f\sin g\sqrt{a} = \frac{(2g - \sin 2g)}{\sin^3 g}[\kappa(1+x)]^{3/2} + \kappa^{3/2}(1+x)^{1/2}$$

or

$$\frac{2g - \sin 2g}{\sin^3 g}(1+x)^{3/2} + (1+x)^{1/2} = \pm m \qquad (\!(5,14)\!)$$

This equation is invalid if $\cos f = 0$ or $\sin g = 0$. If $180° < v_j - v_i < 360°$, then $\cos f$ is negative and m is imaginary. In this case, let

$$M^2 = \frac{\tau^2}{(-2\sqrt{r_i r_j}\cos f)^3}, \quad 1 - 2L = \frac{\sqrt{r_j/r_i} + \sqrt{r_i/r_j}}{2\cos f}$$

Then $a = \dfrac{-2\sqrt{r_i r_j}\cos f\,(L - x)}{\sin^2 g}$ and the corresponding equation becomes

$$\frac{2g - \sin 2g}{\sin^3 g}(L - x)^{3/2} - (L - x)^{1/2} = \pm M \qquad (5,15)$$

For small values of g, the function $\dfrac{2g - \sin 2g}{\sin^3 g} = X(x)$ is of the order of $\dfrac{4}{3}$ + a power series in x. To determine the coefficients of this series, Gauss makes use of the method of undetermined coefficients and the differential relations which the function must satisfy. We have from (5,12):

$$\frac{dx}{dg} = \tfrac{1}{2}\sin g$$

and therefore $\sin^3 g\dfrac{dX}{dg} + 3\sin^2 g\cos g\, X = 2 - 2\cos 2g = 4\sin^2 g$ or $\dfrac{dX}{dg} = \dfrac{4 - 3\cos g\, X}{\sin g}$.

Also

$$\frac{dX}{dx} = \frac{dX}{dg}\frac{dg}{dx} = \frac{8 - 6\cos g\, X}{\sin^2 g} = \frac{4 - 3(1 - 2x)X}{2x(1 - x)}$$

or

$$(2x - 2x^2)\frac{dX}{dx} = 4 - (3 - 6x)X \qquad (5,16)$$

Write

$$X = \sum_0^\infty A_n x^n, \quad \text{and then} \quad \frac{dX}{dx} = \sum_1^\infty n A_n x^{n-1}$$

The differential equation (5,16) becomes

$$(2x - 2x^2)\, n A_n x^{n-1} = 4 - (3 - 6x) A_n x^n.$$

Equating the coefficients of x^n on both sides of this equation, we obtain

$$2n A_n - 2(n - 1)A_{n-1} = -3A_n + 6A_{n-1} \quad \text{or} \quad A_n = \frac{2n + 4}{2n + 3}A_{n-1} \qquad (5,17)$$

The constant term is $A_0 = 4/3$, thus

$$X(x) = \frac{4}{3} + \frac{4\,6}{3\,5}x + \frac{4\,6\,8}{3\,5\,7}x^2 + \ldots.$$

If this is written in the form of a hypergeometric series, $X = \frac{4}{3}F(1,3,2\tfrac{1}{2},x)$, it may then be transformed to the following continued fraction:

$$X = \cfrac{4/3}{1 - \cfrac{\frac{6}{5}x}{1 + \cfrac{\frac{2}{35}x}{1 - \cfrac{\frac{40}{63}x}{1 - \cfrac{\frac{4}{99}x}{1 - \ldots}}}}}$$

where the numerical coefficients are given by the formula

$$\frac{-n(n - 3)}{(2n + 1)(2n + 3)} \text{ if } n \text{ is even}$$

$$\frac{-(n + 5)(n + 2)}{(2n + 1)(2n + 3)} \text{ if } n \text{ is odd}$$

For practical purposes, Gauss writes

$$X = \frac{4/3}{1 - \frac{6}{5}(x - \xi)}$$

where $\xi = x - \dfrac{5}{6} + \dfrac{10}{9X} = \dfrac{2}{35}x^2 + \ldots$, and this may be tabulated as a function of x.

This function ξ is of the 4th order with respect to g, and in the first approximation it will be neglected. Finally, we shall make one more change of variable. Let

$$1 + x = \frac{m^2}{y^2} \text{ and } h = \frac{m^2}{\frac{5}{6} + 1 + \xi} \qquad (5,18)$$

Then if we divide (5,14) by $(1 + x)^{1/2}$ and transpose, our equation may be written in the form

$$\frac{(1+x)}{\frac{3}{4} - \frac{9}{10}(x - \xi)} = \frac{\pm m}{(1+x)^{1/2}} - 1; \text{ then } \frac{m^2}{y^2\left[\frac{3}{4} - \frac{9}{10}\left(\frac{m^2}{y^2} - 1 - \xi\right)\right]} = \frac{m^2}{\frac{9}{10}\left[\left(\frac{5}{6} + 1 + \xi\right)y^2 - m^2\right]} = \frac{1}{\frac{9}{10}\left(\frac{y^2}{h} - 1\right)} = y - 1$$

or

$$y^3 - y^2 - hy - h/9 = 0 \qquad (5,19)$$

With $\xi = 0$ in h, we may solve for y, then x, and then we obtain an approximate value of ξ with which to improve h. The iteration is repeated until y reaches its final value. This process is characteristic of Gauss' method of attacking such numerical problems.

In the other case which we considered, we would let

$$L - x = \frac{M^2}{Y^2} \text{ and } H = \frac{M^2}{L - \frac{5}{6} - \xi}$$

Then the equation becomes $Y^3 + Y^2 - HY + H/9 = 0$. This case will seldom arise in practice.

Now let us return to (5,13) and our solution for a:

$$a = \frac{\kappa}{\sin^2 g}\frac{m^2}{y^2} = \frac{\tau^2}{\kappa^2 y^2 \sin^2 g} = \frac{\tau^2 b^2}{y^2 \kappa^2 r_i r_j \sin^2 f}$$

If we substitute for κ^2, cross-multiply, and cancel, we obtain

$$p = \frac{y^2[r_i r_j \sin(v_j - v_i)]^2}{\tau^2} \qquad (5,20)$$

which shows that we may begin to derive the elements, once we have the solution for y, for

$$e\cos v = \frac{p - r}{r}, \quad e\sin v_i = \frac{e\cos v_i \cos(v_j - v_i) - e\cos v_j}{\sin(v_j - v_i)}$$
$$e\sin v_j = \frac{e\cos v_i - e\cos v_j \cos(v_j - v_i)}{\sin(v_j - v_i)} \qquad (5,21)$$

These enable us to determine e, a, and M or T. The normal vector, $\mathbf{r}_i \times \mathbf{r}_j$, gives the position of the orbit in space. If we recall that $2(A_j - A_i) = k(t_j - t_i)\sqrt{p}$, we may write (5,20) in the form:

$$y = \frac{k(t_j - t_i)\sqrt{p}}{r_i r_j \sin(v_j - v_i)} = \frac{(r_i, r_j)}{[r_i, r_j]} \qquad (5,22)$$

and we see that the unknown, y, to which our solution was eventually reduced is the same as the sector-triangle ratio, $\bar{y}$, in (5,7). From the definition of y, it is apparent that m and $(1 + x)^{1/2}$ are proportional to the area of the sector and the triangle, and $X(1 + x)^{3/2}$ is proportional to their difference, or the area between the chord and the arc.

With this understanding of the meaning of y, we shall return to a consideration of the solution of (5,19). This is an interesting family of curves in the parameter h. The solution may, of course, be tabulated as a function of h. That the cubic equation has only one valid solution may be shown in several ways. First, consider the situation in which t_j approaches t_i; then h approaches zero, and we know from geometrical considerations that the sector-triangle ratio approaches unity. Then (5,19) may be written in the form $y^2(y - 1) = h(y + 1/9)$. If $h = 0$, the equation has the double root $y = 0$ and the single root $y = 1$. The latter is the physical solution. Assuming that we have very small values for h (by definition h is positive), we may write (5,19) in the following form in order to solve by iteration for the roots in the neighborhood of the origin: $y^2 = \frac{h(y + 1/9)}{y - 1}$.
But we see that if h is small, y is small and the denominator is negative, so that these roots are imaginary. The other root may be found by writing the equation in the form: $y = 1 + \frac{h(y + 1/9)}{y^2}$

and this gives the real solution. The analysis may also be made by the more orthodox methods of the Theory of Equations. When tables for the solution are not available, it is best to use Newton's method of approximation:

$$y_{i+1} = y_i + \frac{h/9 + hy_i + y_i^2 - y_i^3}{3y_i^2 - 2y_i - h} = \frac{2y_i^3 - y_i^2 + h/9}{3y_i^2 - 2y_i - h}$$

This will converge more rapidly than the form $y = 1 + \dfrac{h(y + 1/9)}{y^2}$.

Hansen derived an approximate formula which is valid for small values of h. Let $y = 1 + z$, and write ((5,19)) in the form:

$$\frac{y^2(y-1)}{(y+1/9)} = h = \frac{(1+z)^2 z}{z + 10/9}$$

For $(1 + z)^2$, substitute $(1 + 0.9z)(1 + 1.1z)$, which is in error by $0.01z^2$. Then

$$z = \frac{10h/9}{1 + 11z/10} = \cfrac{10h/9}{1 + \cfrac{11h/9}{1 + \cfrac{11h/9}{1 + \cfrac{11h/9}{1 + \dots}}}}$$

Another simple scheme is to compute $y_0 = [6 + 5\sqrt{1 + 44h/9}]/11$, then $y = y_0 - \Delta y$, where Δy is tabulated as in M.N.R.A.S. 90:814.

We may now recapitulate our development thus far. Given three radius vectors at three specified times, we have κ, l, and m known, and we may solve for h, x, and y for the three combinations of the three radius vectors taken in pairs. With the y's known, we may then evaluate the c's. With the c's we are able to solve the components of the fundamental vector equation ((5,6)) for the ρ's, and then $\mathbf{p} - \mathbf{R} = \mathbf{r}$ will give us the three radius vectors which we needed to start with in the first place. Our problem is now to pierce this circuitous functional relationship in some manner. This may be done in several ways, but before attacking this problem it will be advantageous to examine some of the relationships between the c's and our previous results.

Closed expressions for the c's may be obtained in the same manner as we obtained those for f and g. We have $\mathbf{r}_i \times \mathbf{r}_j = \mathbf{A} \times \mathbf{B}\,[(\cos E_i - e)\sin E_j - (\cos E_j - e)\sin E_i]$,

therefore
$$c_1 = \frac{(\cos E_2 - e)\sin E_3 - (\cos E_3 - e)\sin E_2}{(\cos E_1 - e)\sin E_3 - (\cos E_3 - e)\sin E_1}, \quad c_3 = \frac{(\cos E_1 - e)\sin E_2 - (\cos E_2 - e)\sin E_1}{(\cos E_1 - e)\sin E_3 - (\cos E_3 - e)\sin E_1}$$

By suitable substitution, we may obtain for nearly parabolic orbits ((5,23))

$$c_1 = \frac{(\tan\frac{1}{2}v_3 - \tan\frac{1}{2}v_2)(1 + \tan\frac{1}{2}v_3\tan\frac{1}{2}v_2)D_2}{(\tan\frac{1}{2}v_3 - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_3\tan\frac{1}{2}v_1)D_1}, \quad c_3 = \frac{(\tan\frac{1}{2}v_2 - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_2\tan\frac{1}{2}v_1)D_2}{(\tan\frac{1}{2}v_3 - \tan\frac{1}{2}v_1)(1 + \tan\frac{1}{2}v_3\tan\frac{1}{2}v_1)D_3}$$

Furthermore, if we write $\mathbf{r}_1 = f_1\mathbf{r}_2 + g_1\mathbf{r}_2'$ and $\mathbf{r}_3 = f_3\mathbf{r}_2 + g_3\mathbf{r}_2'$, we have

$$\mathbf{r}_1\times\mathbf{r}_2 = -g_1\mathbf{r}_2\times\mathbf{r}_2' = 2[r_1, r_2] \qquad \mathbf{r}_2\times\mathbf{r}_3 = +g_3\mathbf{r}_2\times\mathbf{r}_2' = 2[r_2, r_3] \qquad \mathbf{r}_1\times\mathbf{r}_3 = +g_2\mathbf{r}_2\times\mathbf{r}_2' = 2[r_1, r_3]$$

where $g_2 = f_1 g_3 - f_3 g_1$, and therefore

$$c_1 = \frac{g_3}{g_2}, \quad c_3 = -\frac{g_1}{g_2} \qquad ((5,24))$$

Also $\mathbf{r}_2\times\mathbf{r}_2' = 2\dfrac{dA}{dt} = \dfrac{2(r_2, r_3)}{T_1} = \dfrac{2(r_1, r_3)}{T_2} = \dfrac{2(r_1, r_2)}{T_3}$, therefore

$$g_1 = -\frac{T_3}{y_3}, \quad g_2 = \frac{T_2}{y_2} = f_1\frac{T_1}{y_1} + f_3\frac{T_3}{y_3}, \quad g_3 = \frac{T_1}{y_1} \qquad ((5,25))$$

The equation ((4,22)) may be readily transformed to

$$r_2(1 - f_1) = a(1 - \cos E_1\cos E_2 - \sin E_1 \sin E_2) = a(1 - \cos 2g_3) = 2a\sin^2 g_3 = 2T_3^2/\kappa_3^2 y_3^2$$

or
$$f_1 = 1 - \frac{2T_3^2}{r_2\kappa_3^2 y_3^2}, \quad f_3 = 1 - \frac{2T_1^2}{r_2\kappa_1^2 y_1^2} \qquad ((5,26))$$

These formulas are of more than academic interest; they enable us to determine $\mathbf{r}_0'$ when $\mathbf{r}_0$ and $\mathbf{r}_1$

are given, for f and g now depend only upon known quantities and the sector-triangle ratio of the two radius vectors.

In keeping with the complementary arrangement of the subscripts in this chapter, we have written $T_3 = k(t_2 - t_1)$, $T_2 = k(t_3 - t_1)$, $T_1 = k(t_3 - t_2)$. Then $\tau_1 = -T_3$, $\tau_3 = T_1$, and $\tau_3 - \tau_1 = T_2$. Our former results in equation ((4,4)) are now written

$$f_1 = 1 - \tfrac{1}{2}\mu T_3^2 - \tfrac{1}{2}\mu\sigma T_3^3 + \ldots, \qquad -g_1 = T_3 - \tfrac{1}{6}\mu T_3^3 - \tfrac{1}{4}\mu\sigma T_3^4 + \ldots$$
$$f_3 = 1 - \tfrac{1}{2}\mu T_1^2 + \tfrac{1}{2}\mu\sigma T_1^3 + \ldots, \qquad g_3 = T_1 - \tfrac{1}{6}\mu T_1^3 + \tfrac{1}{4}\mu\sigma T_1^4 + \ldots$$

where μ and σ are computed for t_2. Then

$$g_2 = T_2 - \tfrac{1}{6}\mu T_2^3 - \tfrac{1}{4}\mu\sigma T_2^3(T_3 - T_1) + \ldots$$
$$\left.\begin{aligned} c_1 &= \frac{T_1}{T_2}\left[1 + \frac{1}{6}\mu(T_2^2 - T_1^2) + \frac{1}{4}\mu\sigma T_3(T_2T_3 - T_1^2) + \ldots\right] \\ c_3 &= \frac{T_3}{T_2}\left[1 + \frac{1}{6}\mu(T_2^2 - T_3^2) - \frac{1}{4}\mu\sigma T_1(T_2T_1 - T_3^2) + \ldots\right] \end{aligned}\right\} \tag{(5,27)}$$

A more accurate expression for the c's was given by Gibbs in the Memoirs of the National Academy of Science, 1888. Let

$$\mathbf{r} = \mathbf{R}_0 + \mathbf{R}_1 T + \mathbf{R}_2 T^2 + \mathbf{R}_3 T^3 + \mathbf{R}_4 T^4 + \ldots \tag{(5,28)}$$

Differentiate this expression twice and impose the law of gravitation.

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3} = 2\mathbf{R}_2 + 6\mathbf{R}_3 T + 12\mathbf{R}_4 T^2 + \ldots$$

Then
$$(1) = \mathbf{r}_1 = \mathbf{R}_0 - \mathbf{R}_1 T_3 + \mathbf{R}_2 T_3^2 - \mathbf{R}_3 T_3^3 + \mathbf{R}_4 T_3^4 - \ldots$$
$$(2) = \mathbf{r}_2 = \mathbf{R}_0$$
$$(3) = \mathbf{r}_3 = \mathbf{R}_0 + \mathbf{R}_1 T_1 + \mathbf{R}_2 T_1^2 + \mathbf{R}_3 T_1^3 + \mathbf{R}_4 T_1^4 + \ldots$$
$$(4) = -\frac{\mathbf{r}_1}{r_1^3} = 2\mathbf{R}_2 - 6\mathbf{R}_3 T_3 + 12\mathbf{R}_4 T_3^2 - \ldots$$
$$(5) = -\frac{\mathbf{r}_2}{r_2^3} = 2\mathbf{R}_2$$
$$(6) = -\frac{\mathbf{r}_3}{r_3^3} = 2\mathbf{R}_2 + 6\mathbf{R}_3 T_1 + 12\mathbf{R}_4 T_1^2 + \ldots$$

From these we obtain

$$(7) = T_1^2(4) - (T_1^2 - T_3^2)(5) - T_3^2(6) = -T_1^2\frac{\mathbf{r}_1}{r_1^3} + (T_1^2 - T_3^2)\frac{\mathbf{r}_2}{r_2^3} + T_3^2\frac{\mathbf{r}_3}{r_3^3} = -6\,T_1T_2T_3\mathbf{R}_3$$
$$(8) = T_1(4) - T_2(5) + T_3(6) = -T_1\frac{\mathbf{r}_1}{r_1^3} + T_2\frac{\mathbf{r}_2}{r_2^3} - T_3\frac{\mathbf{r}_3}{r_3^3} = 12\,T_1T_2T_3\mathbf{R}_4$$
$$(9) = T_1(1) - T_2(2) + T_3(3) = T_1\mathbf{r}_1 - T_2\mathbf{r}_2 + T_3\mathbf{r}_3 = T_1T_2T_3\mathbf{R}_2 + T_1T_3(T_1^2 - T_3^2)\mathbf{R}_3 + T_1T_3(T_1^3 + T_3^3)\mathbf{R}_4 + \ldots$$

In this last expression, eliminate the **R**'s by substitution from (5), (7), and (8), and collect terms. The result is

$$T_1\left[1 + \frac{(T_2^2 + T_1T_3 - T_1^2)}{12\,r_1^3}\right]\mathbf{r}_1 - T_2\left[1 - \frac{(T_1^2 + 3\,T_1T_3 + T_3^2)}{12\,r_2^3}\right]\mathbf{r}_2 + T_3\left[1 + \frac{(T_2^2 + T_1T_3 - T_3^2)}{12\,r_3^3}\right]\mathbf{r}_3 = 0$$

or if this expression is written in the form of ((5,4)), then

$$c_1 = \frac{T_1(1 + B_1 r_1^{-3})}{T_2(1 - B_2 r_2^{-3})}, \quad c_3 = \frac{T_3(1 + B_3 r_3^{-3})}{T_2(1 - B_2 r_2^{-3})} \tag{(5,29)}$$

where
$$B_1 = (T_1T_3 + T_2(T_3 - T_1))/12 = (mn + (2n - 1))T_2^2/12$$
$$B_2 = (T_1T_3 + T_2^2)/12 = (mn + 1)T_2^2/12$$
$$B_3 = (T_1T_3 - T_2(T_3 - T_1))/12 = (mn - (2n - 1))T_2^2/12.$$

Here $n = T_3/T_2$ and $m = 1 - n = T_1/T_2$; they have been introduced because this operation may be looked upon as an interpolation for $\mathbf{r}_2$ from $\mathbf{r}_1$ and $\mathbf{r}_3$ by an extension of Everett's formula in which

the second and higher order differences are expressed in terms of the function in the manner of a "throwback", and the c's become modified interpolating factors. These expressions are accurate to the third order, because our original equation ((5,28)) was accurate to the fourth order, but the expression (9) from which the **R**'s were finally eliminated was essentially a first order difference equation, thus reducing the accuracy of the result to the third order. In the special case of equal time intervals, it is evident that all the odd orders of **R** vanish from expression (9), and therefore equation ((5,28)) might equally well have been carried to the fifth order, and the resulting formulas for the c's would be the same. They are therefore in this case of equal intervals accurate to the fourth order.

In the Bulletin de l'Academie Polonaise des Sciences et des Lettres, 1936, or the Cracow Observatory Reprint No. 13, Koziel has investigated the accuracy of various expressions for the triangle ratios, and he gives a correction term to the Gibbs formulas which makes them accurate to the fourth order. This is given here without proof.

$$c_1 = m\,\frac{(1 + B_1 r_1^{-3} - nC)}{1 - B_2 r_2^{-3}}, \qquad c_3 = n\,\frac{(1 + B_3 r_3^{-3} + mC)}{1 - B_2 r_2^{-3}} \tag{(5,30)}$$

where $C = \dfrac{(2 + mn)(m - n)}{60}\left[\dfrac{T_2^2}{6 r_2^6} - \dfrac{1}{mn}\left(\dfrac{m}{r_1^3} - \dfrac{1}{r_2^3} + \dfrac{n}{r_3^3}\right)\right] T_2^2.$

Now we are prepared to return to the main problem. Operate upon both sides of ((5,6)) by $\cdot(\mathbf{p}_1^* \times \mathbf{p}_3^*)$, and we get

$$[\mathbf{p}_1^* \cdot \mathbf{p}_2^* \times \mathbf{p}_3^*]\rho_2 = c_1[\mathbf{R}_1 \cdot \mathbf{p}_1^* \times \mathbf{p}_3^*] - [\mathbf{R}_2 \cdot \mathbf{p}_1^* \times \mathbf{p}_3^*] + c_3[\mathbf{R}_3 \cdot \mathbf{p}_1^* \times \mathbf{p}_3^*] \tag{(5,31)}$$

Taking into account only the first two terms of the series expansions ((5,27)) for the c's, we have

$$\begin{aligned} c_1 &= c_1^\circ + \nu_1/r_2^3 = \frac{T_1}{T_2} + \frac{1}{6}\frac{T_1}{T_2}\left(1 - \frac{T_1^2}{T_2^2}\right)\frac{T_2^2}{r_2^3} = m + \frac{m(1 - m^2)}{6}\,\Delta^{II} \\ c_3 &= c_3^\circ + \nu_3/r_2^3 = \frac{T_3}{T_2} + \frac{1}{6}\frac{T_3}{T_2}\left(1 - \frac{T_3^2}{T_2^2}\right)\frac{T_2^2}{r_2^3} = n + \frac{n(1 - n^2)}{6}\,\Delta^{II} \end{aligned} \tag{(5,32)}$$

We see that the ν's are proportional to the Everett second difference coefficients, and that both second differences have been set equal to T_2^2/r_2^3, so that no third order effects are included. Thus ((5,31)) becomes $E\,\rho_2 = (c_1^\circ F_1 - F_2 + c_3^\circ F_3) + (\nu_1 F_1 + \nu_3 F_3)/r_2^3$, or $\rho_2 = A + B/r_2^3$. ((5,33))

The "triangle" equation is $r^2 = \rho^2 - 2(\mathbf{p}^* \cdot \mathbf{R})\rho + R^2$. These two equations in the two unknowns, ρ_2 and r_2, are of the same form as we derived in the proof of Lambert's theorem and as we had at the corresponding stage in the solution by the method of La Place. After these unknowns are determined, we may return to the fundamental equation ((5,6)), eliminate ρ_2, solve for $c_1\rho_1$, $c_3\rho_3$, and then ρ_1 and ρ_3.

Our problem would now be solved except for one defect. Our observations are all exactly satisfied since our values of the ρ's have been derived directly from the equations which express the geometrical conditions, but our c's were obtained from approximate formulas, and therefore there is no guarantee that the motion of the object is in strict accordance with the law of gravity. Let us ameliorate this defect in the following way. With the ρ's we now have, we may obtain the corresponding **r**'s by means of the equation $\mathbf{r} = \rho\mathbf{p}^* - \mathbf{R}$; and with these we then use a more accurate formula to compute the c's which correspond to our adopted solution at the present stage. If our former solution is in error at all, the trouble must be with the ν's, since c_1° and c_3° are fixed. Therefore compute new values of the ν's from $\nu_1 = r_2^3(c_1 - c_1^\circ)$, and use these to repeat the solution.

This process must be repeated successively until the values of the ν's no longer change. Then the dynamical conditions will be satisfied to the extent that they are imposed by the formula we used for the c's. The reader may well wonder whether or not this process will converge to a definite solution. In certain cases it may not, and the computer must judge this by an examination of the successive values of the ν's which he obtains; but the cause of the difficulty, if it exists, is almost certainly to be found in the inadequacy of the observations to permit a determinate solution. In general, this method of solution will readily converge, due mainly to the fact that the c's are relatively insensitive to the r's and therefore a large change in the latter, in going from one approximation to the next, will not produce a correspondingly large change in the former, so that

the next solution for the ρ's will not differ very greatly from the preceding one. This is just another way of stating that the differential coefficient must be considerably less than unity if this iterative process is to converge nicely.

This method is essentially the modification of Gauss' method which was given by Merton in M.N.R.A.S. vol. 85, p. 693 and vol. 89, p. 451. Another valuable discussion of this subject by Innes will be found in the same publication, vol.89, p.422. We shall not develop the details of this method to any greater extent because in practice it is possible to take the same advantage of the principle of using the ratios of the direction cosines that we did in the method of La Place.

Operate upon the fundamental equation ((5,6)) successively with $\cdot(\mathbf{p}_2^* \times \mathbf{p}_3^*)$ and $\cdot(\mathbf{p}_1^* \times \mathbf{p}_2^*)$. We obtain

$$\begin{aligned} c_1[\mathbf{p}_1^*\cdot\mathbf{p}_2^*\times\mathbf{p}_3^*]\rho_1 &= c_1[\mathbf{R}_1\cdot\mathbf{p}_2^*\times\mathbf{p}_3^*] - [\mathbf{R}_2\cdot\mathbf{p}_2^*\times\mathbf{p}_3^*] + c_3[\mathbf{R}_3\cdot\mathbf{p}_2^*\times\mathbf{p}_3^*] \\ c_3[\mathbf{p}_1^*\cdot\mathbf{p}_2^*\times\mathbf{p}_3^*]\rho_3 &= c_1[\mathbf{R}_1\cdot\mathbf{p}_1^*\times\mathbf{p}_2^*] - [\mathbf{R}_2\cdot\mathbf{p}_1^*\times\mathbf{p}_2^*] + c_3[\mathbf{R}_3\cdot\mathbf{p}_1^*\times\mathbf{p}_2^*] \end{aligned} \qquad ((5,34))$$

These equations give the solutions for ρ_1 and ρ_3 directly, as soon as the c's are known. Furthermore, they satisfy all of the geometrical conditions exactly, so that there are no residuals for any of the observations. This procedure also falls in the category with those methods in which the geometrical conditions are always satisfied but the dynamical conditions are not satisfied until the c's have converged by successive approximations to their final values. As we have already demonstrated in the La Placian method, these triple scalar products may be reduced to second order determinants if we write

$$\begin{aligned} &U = \tan\alpha,\ V = \sec\alpha\tan\delta,\ P = Y - UX,\ Q = Z - VX \\ &y = Ux - P,\ z = Vx - Q,\ r^2 = (1 + U^2 + V^2)x^2 - 2(UP + VQ)x + (P^2 + Q^2) \end{aligned} \qquad ((5,35))$$

Since $\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3$, we have

$$y_2 = U_2(c_1x_1 + c_3x_3) - P_2 = c_1(U_1x_1 - P_1) + c_3(U_3x_3 - P_3)$$

$$z_2 = V_2(c_1x_1 + c_3x_3) - Q_2 = c_1(V_1x_1 - Q_1) + c_3(V_3x_3 - Q_3).$$

After transposing and collecting terms, we have

$$\begin{aligned} c_1(U_1 - U_2)x_1 + c_3(U_3 - U_2)x_3 &= c_1P_1 - P_2 + c_3P_3 = P \\ c_1(V_1 - V_2)x_1 + c_3(V_3 - V_2)x_3 &= c_1Q_1 - Q_2 + c_3Q_3 = Q \end{aligned} \qquad ((5,36))$$

or if $(U_1 - U_2)(V_3 - V_2) - (U_3 - U_2)(V_1 - V_2) = D,$

then

$$\begin{aligned} c_1D\,x_1 &= P(V_3 - V_2) - Q(U_3 - U_2) \\ c_3D\,x_3 &= Q(U_1 - U_2) - P(V_1 - V_2) \end{aligned} \qquad ((5,37))$$

All the conditions of the general solution are contained in this simple pair of equations. The c's provide that the heliocentric, rectangular coordinates of the middle place are derived from those of the first and third places according to the law of gravitation and that the three radius vectors are coplanar; while the U, V, P, and Q's insure that all three observations are exactly represented. As indicated before, these equations may be solved for the coordinates by using approximate values for the c's, and then the coordinates give more accurate values of the c's with which to repeat the solution until it is final. However, it is possible to develop expressions for the c's which are more convenient for the present purpose than those already given above.

As we did in equation ((5,28)), let us write

$$\mathbf{r} = \mathbf{R}_0 + \mathbf{R}_1T + \mathbf{R}_2T^2 + \mathbf{R}_3T^3 + \mathbf{R}_4T^4 + \dots \qquad ((5,38))$$

This time let us agree to count T in units of $T_2 = k(t_3 - t_1)$; and set $T = -1, 0, +1$.

$$\begin{aligned} \mathbf{r}_{-1} &= \mathbf{R}_0 - \mathbf{R}_1 + \mathbf{R}_2 - \mathbf{R}_3 + \mathbf{R}_4 - \dots \\ -2\mathbf{r}_0 &= -2\mathbf{R}_0 \\ \mathbf{r}_{+1} &= \mathbf{R}_0 + \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_4 + \dots \\ \Delta^{II}\mathbf{r}_0 &= 2\mathbf{R}_2 + 2\mathbf{R}_4 + \dots \end{aligned}$$

Similarly $\Delta^{iv}\mathbf{r}_0 = +24\mathbf{R}_4 + \dots$

Now by Taylor's series and the law of gravitation

$$R_2 = \tfrac{1}{2}T_2^2\frac{d^2 r_0}{dt^2} = -\tfrac{1}{2}T_2^2\frac{r_0}{r_0^3}, \quad \text{and therefore} \quad \Delta^{ii} r_0 = -\frac{T_2^2 r_0}{r_0^3} + \frac{1}{12}\Delta^{iv} r_0 \qquad ((5{,}39))$$

Also

$$\begin{aligned}
\Delta^{iv} r_0 &= \Delta^{ii} r_1 - 2\Delta^{ii} r_0 + \Delta^{ii} r_{-1} \\
&= \frac{-T_2^2 r_1}{[r_0 + (r_1 - r_0)]^3} + \frac{2T_2^2 r_0}{r_0^3} - \frac{T_2^2 r_{-1}}{[r_0 + (r_{-1} - r_0)]^3} + \frac{1}{12}\Delta^{vi} r_0 \\
&= -\frac{T_2^2}{r_0^3}(r_1 - 2r_0 + r_{-1}) + 3[(r_1 - r_0)r_1 - (r_0 - r_{-1})r_{-1}]\frac{T_2^2}{r_0^4} + \frac{1}{12}\Delta^{vi} r_0 \\
&= +\frac{T_2^4}{r_0^6} r_0 \qquad ((5{,}40))
\end{aligned}$$

where we have neglected the sixth difference and also the term factored by 3, which is of the 4th order, but which vanishes for circular orbits or in the neighborhood of perihelion and aphelion. These expressions for the differences enable us to derive r_2 from r_1 and r_3 by interpolation with Everett's formula:

$$\begin{aligned}
r_2 &= m\, r_1 + \frac{m(1-m^2)}{6}K_1\left(1 - \frac{1}{12}K_1\right) r_1 + \frac{m(1-m^2)(4-m^2)}{120}K_1^2 r_1 \\
&\quad + n\, r_3 + \frac{n(1-n^2)}{6}K_3\left(1 - \frac{1}{12}K_3\right) r_3 + \frac{n(1-n^2)(4-n^2)}{120}K_3^2 r_3 \qquad ((5{,}41)) \\
&= \left\{m + \frac{m(1-m^2)}{6}K_1\left[1 + \frac{7-3m^2}{60}K_1\right]\right\} r_1 + \left\{n + \frac{n(1-n^2)}{6}K_3\left[1 + \frac{7-3n^2}{60}K_3\right]\right\} r_3
\end{aligned}$$

where $K_i = T_2^2/r_i^3$, and the $\{\}$'s are expressions for the c's. These are especially convenient to use in practical computations for the Everett coefficients may be taken from tables, r_i^2 is a simple function of only one of the two independent variables, r_i^{-3} may be taken from Table X of Planetary Coordinates, the remaining term is tabulated in the adjoining small table, and all the terms of the formula are positive and additive. If we assume some reasonable values for r_1^2 and r_3^2, say 8.0 or 10.0 for a minor planet or 2.0 or 4.0 for a comet (unless it appears to be closer to the Sun as shown by Lambert's theorem), then these permit the evaluation of the c's, and then x_1 and x_3. If these give values of r_1^2 and r_3^2 which differ widely from the original assumptions, the computer may make new assumptions, or he may make several assumptions to begin with and then choose the best one on the basis of how well r_i^2 reproduces itself. Once values reasonably near the solution are obtained, the resulting r^2's are used to recompute the c's, x's, etc. until the solution is final.

m	$\frac{7-3m^2}{60}$	
0.0	0.1167	
		- 5
0.1	0.1162	
		-15
0.2	0.1147	
		-25
0.3	0.1122	
		-35
0.4	0.1087	
		-45
0.5	0.1042	
		-55
0.6	0.0987	
		-65
0.7	0.0922	
		-75
0.8	0.0847	
		-85
0.9	0.0762	
		-95
1.0	0.0667	

This iterative process takes account of the effect on the value of r produced by the successive changes in the value of x only in the next cycle of computations, not in the same cycle. Due to the fact that the expressions for the c's are now simple functions of only one independent variable, it is possible to take this effect into account directly and thus produce a much more rapid convergence of the values of the unknowns to the ultimate solution. From the equation for r^2 in ((5,35)):

$$dr = \frac{(1 + U^2 + V^2)x - (UP + VQ)}{r}\,dx$$

The effect upon c_1 of a variation of r_1 is given by

$$dc_1 = -3\frac{m(1-m^2)}{6}\frac{K_1}{r_1}dr_1 = -3\frac{m(1-m^2)}{6}K_1\left[\frac{(1 + U_1^2 + V_1^2)x_1 - (U_1P_1 + V_1Q_1)}{r_1^2}\right]dx_1 = -C_1 dx_1$$

and similarly for dc_3. Now the equations to be solved, corresponding to ((5,36)), are

$$(c_1 + dc_1)(U_1 - U_2)(x_1 + dx_1) + (c_3 + dc_3)(U_3 - U_2)(x_3 + dx_3) = (c_1 + dc_1)P_1 - P_2 + (c_3 + dc_3)P_3$$

$$(c_1 + dc_1)(V_1 - V_2)(x_1 + dx_1) + (c_3 + dc_3)(V_3 - V_2)(x_3 + dx_3) = (c_1 + dc_1)Q_1 - Q_2 + (c_3 + dc_3)Q_3$$

Since x_1 and x_3 are our approximate values of these quantities, they are fixed and the unknowns are now the differential corrections, dx_1 and dx_3, which are to be added to x_1 and x_3. These will be found by transforming the equations to

$$\begin{aligned}&[(U_1 - U_2)(c_1 - C_1x_1) + P_1C_1]\,dx_1 + [(U_3 - U_2)(c_3 - C_3x_3) + P_3C_3]\,dx_3 = P - (U_1 - U_2)c_1x_1 - (U_3 - U_2)c_3x_3\\ &[(V_1 - V_2)(c_1 - C_1x_1) + Q_1C_1]\,dx_1 + [(V_3 - V_2)(c_3 - C_3x_3) + Q_3C_3]\,dx_3 = Q - (V_1 - V_2)c_1x_1 - (V_3 - V_2)c_3x_3\end{aligned} \tag{5,42}$$

Terms of the second order have been neglected in these equations. Therefore the corrected values of x_1 and x_3 must be tested to see whether the right hand members are reduced to zero or whether still further corrections must be determined. In the latter case, it is not necessary to recompute the coefficients of the left hand members of the equations unless the previous corrections have been large. Before the solution can be considered final, it is necessary to correct all the times of observation for "light time" or planetary aberration. It is also advisable to check the final solution by a recomputation with closed expressions for the c's, based on the sector-triangle ratios.

The components of $\mathbf{r}_1$ and $\mathbf{r}_3$ are now the six constants of integration or elements of the orbit. If we wish to transform to the vectorial constants or the elliptical elements, we have derived all the relationships we need, and the order of the computation is shown in the following example.

This completes the presentation of the Gaussian method of determining a preliminary orbit. The careful reader will perceive that it is not mandatory to follow any single, specific, prescribed procedure in order to obtain the solution, and personal preferences or expediency may dictate one course or another. Some may prefer to avoid the separation of the sky into three regions for the purpose of reducing to second order determinants and instead evaluate the triple scalar products directly. It is also possible to set up procedures which make hybrid combinations of features of the La Placian and Gaussian methods. Both methods begin essentially with the dynamical, approximate equation $\rho = A + B/r^3$ and the "triangle" equation. The La Placian method guarantees the exact representation of only the middle observation and there remain four degrees of variability. The importance of this will become evident in the next chapter. The Gaussian method as here presented represents all three observations exactly and if the final result is made to depend upon sufficiently accurate formulas for the c's, there is no possibility of further improvement without using more observations. In this respect, the steps which have been described in the Gaussian method carry the solution to a greater state of completion than do those which have been described for the La Placian method. The comparison will be clarified by the examples which follow. We shall now use the Gaussian method to repeat the solution which has been determined in Chapter 4.

1935 UT	Aug. 30.0006	Sept. 2.9067	Sept. 6.9351
JD	44.5006	48.4067	52.4351
	-0.9217386	-0.9460249	-0.9667071
R	+0.3782763	+0.3214131	+0.2612860
	+0.1640270	+0.1393582	+0.1132835
U	-0.2395896	-0.2507026	-0.2625363
V	-0.0663341	-0.0813223	-0.0971067
P	+0.1574373	+0.0842422	+0.0074903
Q	+0.1028843	+0.0624253	+0.0194098
S	1.0304384	1.0341495	1.0384389
$r^2 =$	$+1.0618034\,x^2$	$+1.0694651\,x^2$	$+1.0783550\,x^2$
	$+0.0890902\,x$	$+0.0523926\,x$	$+0.0077026\,x$
	+0.0353717	+0.0109937	+0.0004328

$U_1 - U_2 = +0.0111130$, $U_3 - U_2 = -0.0118337$, $P = -0.0006157$, -6089, -6080, -6079
$V_1 - V_2 = +0.0149882$, $V_3 - V_2 = -0.0157844$, $Q = -0.0006296$, -6247, -6240, -6239

$D = +0.0000019538$

The successive, computed values of P and Q are placed here for convenience in the arrangement of the computation.

T_2	0.1364905	T_2^2	0.0186297
m	0.5077068	n	0.4922932
E_0	0.06280	E_1	0.06216
	0.1038		0.1046
r^2	9.0	r^{-3}	0.0370370
c_1	0.5077501	c_3	0.4923361
c_1D	+0.0000009920	c_3D	+0.0000009619
x_1	+2.2862	x_3	+2.3199
r^2	5.789	r^2	5.828
K	0.0013375	K	0.0013241

c	0.5077908	0.4923755
c D	+0.0000009921	+0.0000009620
x	+2.2363	+2.2703
r^2	5.54472	5.57604
K	0.0014269	0.0014149
	0.5077964	0.4923812
	+0.0000009921	+0.0000009620
	+2.2303	+2.2644
	5.51573	5.54715
	0.0014381	0.0014259

c	0.5077971	0.4923818
c D	0.0000009921	0.0000009620
	+2.2299000	+2.2640000
$\mathbf{r}$	-0.6916981	-0.6018725
	-0.2508027	-0.2392594
r^2	5.51380	5.54519
K	0.0014389	0.0014267

The correction for "light time" does not change T_2, but the corrected values of the observed times are JD 44.4928, 48.3989, 52.4273. The next step would be to deduce the usual elements for purposes of identification, and to compute an ephemeris to facilitate further observations. The computation of the elements will be illustrated later. One of the most expeditious and accurate methods of computing an ephemeris is based upon equation ((5,29)) when $n = \frac{1}{2}$ and the equation is written in the following form:

$$\mathbf{r}_3 = \frac{(2 - 10\,T^2/12\,r_2^3)\,\mathbf{r}_2 - (1 + T^2/12\,r_1^3)\,\mathbf{r}_1}{(1 + T^2/12\,r_3^3)} \qquad ((5,43))$$

Here $2T = T_2$, i.e. T is the interval of the ephemeris expressed in units of $1/k$ mean solar days. If we have any two sets of values of the components of $\mathbf{r}_1$ and $\mathbf{r}_2$ with which to start a table of the coordinates, we compute an auxiliary column of $10\,T^2/12\,r^3$, extrapolate the denominator, and extend the table one step. This is a simple routine computation with a calculating machine: to fill a position in the table, the closest adjoining position is multiplied by 2 and then (with due caution for the decimal place) all the other terms are subtracted, and there is a final, automatic division. The only other keyboard settings are the value in the second closest adjoining position and the divisor. Then one more value in the auxiliary column is determined and the process is repeated. The table may be extended in either direction with equal facility.

In the present case, we may obtain our starting coordinates in the table from the solution we have just derived if we use equation ((5,41)) with n = +0.5050350 and n = +1.5132901. The complete computation follows:

JD	48.5			56.5	
m	0.4949650	n	0.5050350	-0.5132901	+1.5132901
E_0	0.06270	E_1	0.06270	-0.06301	-0.32537
	0.1045		0.1039	0.1035	0.0022
c	0.4950546		0.5051245	-0.5133808	+1.5128259

JD	x	y	z	r^2	$T^2/12\,r^3$	
48.5	+2.24752	-0.64645	-0.24502	5.52928	0.0001214	
						11
56.5	+2.28025	-0.55542	-0.23320	5.56241	0.0001203	
						12
64.5	+2.30969	-0.46359	-0.22104	5.59844	0.0001191	
						12
72.5	+2.33583	-0.37110	-0.20856	5.63731	0.0001179	
						13
80.5	+2.35867	-0.27808	-0.19578	5.67898	0.0001166	
						13
88.5	+2.37821	-0.18467	-0.18273	5.72338	0.0001153	
						14
96.5	+2.39446	-0.09100	-0.16943	5.77043	0.0001139	
104.5	+2.40744	+0.00279	-0.15590			

JD	x + X	y + Y	z + Z	ρ	$\tan\alpha$	$\sin\delta$	α		δ	
48.5	+1.30099	-0.32643	-0.10622	1.34552	-0.2509	-0.0789	$23^h\ 03^m.7$	51	$-4°\ 32$	101
56.5	+1.29730	-0.35603	-0.14672	1.35324	-0.2744	-0.1084	22 58.6	47	- 6 13	99
64.5	+1.30848	-0.38847	-0.18846	1.37788	-0.2969	-0.1368	22 53.9	40	- 7 52	89
72.5	+1.33501	-0.42173	-0.23052	1.41889	-0.3159	-0.1625	22 49.9	28	- 9 21	76
80.5	+1.37711	-0.45351	-0.27187	1.47513	-0.3293	-0.1843	22 47.1	13	-10 37	61
88.5	+1.43445	-0.48150	-0.31147	1.54483	-0.3357	-0.2016	22 45.8	$\bar{3}$	-11 38	44
96.5	+1.50641	-0.50362	-0.34840	1.62613	-0.3343	-0.2143	22 46.1	18	-12 22	29
104.5	+1.59216	-0.51775	-0.38168	1.71718	-0.3252	-0.2223	22 47.9		-12 51	

So long as the arc does not become too long, there is no need to change the computing procedure when the orbit is to be improved by the use of more observations. We shall now use such preliminary data as are supplied by the ephemeris computations, and derive the orbit based on the first, fourth, and fifth observations of page 24. This time we shall employ the equations ((5,42)) after the first step, and then use the Gibbs formulas for the c's.

1935 UT	Aug. 30.0006	Sept. 23.8717	Oct. 21.8510
JD	44.5006	69.3717	97.3510
	-0.9217386	-1.0032412	-0.8811272
R	+0.3782763	-0.0014225	-0.4245110
	+0.1640270	-0.0006612	-0.1841615
U	-0.2395896	-0.3103097	-0.3461972
V	-0.0663341	-0.1630973	-0.2432054
P	+0.1574373	-0.3127380	-0.729554
Q	+0.1028843	-0.1642871	-0.39845 4
S	1.0304384	1.0596664	1.0858183
r^2 =	$+1.0618034\ x^2$	$+1.1228928\ x^2$	$+1.1790014\ x^2$
	+0.0890902 x	-0.2476808 x	-0.6989532 x
	+0.0353717	+0.1247953	+0.6910177

$U_1 - U_2 = +0.0707201$ $U_3 - U_2 = -0.0358875$ $P = +0.0507092,\ +0.0512758,\ +0.0513337$
$V_1 - V_2 = +0.0967632$ $V_3 - V_2 = -0.0801081$ $Q = +0.0301890,\ +0.0304781,\ +0.0305074$

$D = -0.00219266$

T_2	0.9091130	T_2^2	0.8264864
m	0.5293971	n	0.4706029
E_0	0.06350	E_1	0.06106
	0.1027		0.1056
r^2	5.622		5.776
K	0.062001		0.059538
c	0.5333592		0.4742611
c D	-0.00116948		-0.00103989

x	+2.547123	+2.665489
r^2	7.151101	7.204565
K	0.043219	0.042739
c	0.5321537	0.4732243
C	+0.0031651	+0.0030352

$+0.0375621\ dx_1 - 0.0189068\ dx_3 = +0.0006851$ (-0.0000149)
$+0.0510384\ dx_1 - 0.0384704\ dx_3 = +0.0003657$ (-0.0000024)
-0.000480056

$+0.040499 = dx_1$ $+0.044224 = dx_3$
(-0.001100) (-0.001396)

x	+2.587622	+2.659315	+2.709713
ρ	1.7166	1.7549	1.9855
m, $T_2^2/12$, n	0.5293952	0.0688739	0.4706048
B	0.0131098	0.0860329	0.0212081
r^2	7.3755135	7.4071828	7.4539254
r^{-3}	0.0499243	0.0496044	0.0491386
c_1, div. c_3	0.5320121	0.9957324	0.4731143

With these values of the c's, we obtain the right hand members of the equations shown in ( ) and the final corrections to x_1 and x_3. It is, of course, also possible to proceed alternatively by repeated substitutions of the successive values of the c's into ((5,37)) until the solution converges.

	$\mathbf{r}_1$		$\mathbf{r}_3$		$\mathbf{r}_1 \times \mathbf{r}_3$		$\mathbf{R}$
	+2.5865220		+2.7083170		+0.1451253		+0.0921503
	-0.7771411		-0.2080570		-0.0702546		+0.3548640
	-0.2744589		-0.2602209		+1.5666004		+0.9303655
					1.5748758		
r^2	7.3693720		7.4459836	sin i	0.3666335	i	$21^\circ.5081$
r	2.7146587		2.7287330	tan Ω	-0.2596778	Ω	165.4431
κ^2	29.2916198	$r_1 r_3$	7.4075788	r sin i	0.9952848		
κ	5.4121733	cosΔv	+0.9771386	sin u	+0.0575606	u	176.7002
m	0.0052134	sinΔv	+0.2126033	tan v	+0.117710	v	6.7168
1 + 5/6	0.8362174					ω	169.9834
h	0.0062345	x	0.002258				
y	1.0068752	ξ	0.0000003		$\mathbf{V}$		
					+0.7853010		
p	3.0423432				+2.5982222		
a	3.0879604	P	5.4263471		+0.0437699		
$\sqrt{a}$	1.7572593	n	$0^\circ.1816343$				
e cos v	+0.1207093		+0.1149289		$\mathbf{A}$		$\mathbf{B}$
e sin v	+0.0142087		+0.0395471		+2.8176008		+1.2219901
e	0.1215427	cos ϕ	0.9925862		-1.2234548		+2.8109087
e°	$6^\circ.96388$				-0.3158804		+0.0128543
e(1 + e cos v)	0.1362140		0.1355115	a^2	9.5354963	e	0.1215424
cos E	+0.9946256		+0.9571256	a	3.0879599	e°	$6^\circ.96387$
(cos E - e)	+0.8730829		+0.8355829	P	5.4263459	n	0.18163431
sin E	+0.103538		+0.289672				
tan$\frac{1}{2}$E	+0.0519086		+0.1480090	Epoch	1935 July 17.0 UT		
E	$5^\circ.94296$		$16^\circ.83833$		= JD 2428000.5		
M	5.22193		14.82109		M = $357^\circ.23171$		

Since $\rho_1 = 1.7155$ and $\rho_3 = 1.9840$, the corrected times of observation are JD 44.4907 and 97.3396. Since sin i is so large, it is possible to use the formula involving csc i; otherwise it would be necessary to determine ω by means of $\mathbf{N}$. The elements may also be determined after $\mathbf{A}$ and $\mathbf{B}$ are known.

Several topics which depend upon the relationship of two positions in the orbit are closely associated with the results we have obtained in the development of the Gaussian method. If we write

$$e \cos G = \cos h, \qquad 2g = c - d, \qquad 2h = c + d,$$

then

$$r_i + r_j = 2a(1 - \cos g \cos h) = a(2 - \cos c - \cos d).$$

Also let $\mathbf{S}$ be the chord joining $\mathbf{r}_i$ and $\mathbf{r}_j$; then by the law of cosines

$$\begin{aligned} \mathbf{S}^2 &= r_i^2 + r_j^2 - 2 r_i r_j \cos 2f = (r_i + r_j)^2 - 4 r_i r_j \cos^2 f \\ &= 4a^2(1 - \cos g \cosh)^2 - 4a^2(\cos g \cos h)^2 = (2a \sin g \sin h)^2 \\ &= a^2(\cos d - \cos c)^2 \end{aligned}$$

Then

$$\begin{aligned} (r_i + r_j + S)/a &= 2(1 - \cos c) = 4 \sin^2 \tfrac{1}{2} c \\ (r_i + r_j - S)/a &= 2(1 - \cos d) = 4 \sin^2 \tfrac{1}{2} d \end{aligned} \qquad ((5,44))$$

These results have come from equations which express only the geometrical properties of an ellipse. As before, we impose the dynamical conditions by means of Kepler's equation. This becomes

$$k(t_j - t_i)a^{-3/2} = 2g - 2 \sin g \cos h = (c - d) - (\sin c - \sin d) = (c - \sin c) - (d - \sin d).$$

From this we may write

$$6k(t_j - t_i) = \frac{3}{4}\left(\frac{c - \sin c}{\sin^3\frac{1}{2}c}\right)(4a\sin^2\tfrac{1}{2}c)^{3/2} - \frac{3}{4}\left(\frac{d - \sin d}{\sin^3\frac{1}{2}d}\right)(4a\sin^2\tfrac{1}{2}d)^{3/2}$$
$$= Q(c)\,(r_i + r_j + S)^{3/2} - Q(d)\,(r_i + r_j - S)^{3/2} \qquad ((5,45))$$

which is known as Lambert's theorem on the motion in a conic section. This is essentially the same as the equation ((5,11)) which Gauss used to determine the sector-triangle ratio, except that g has been eliminated in favor of S. Lambert was attempting to improve upon the then current practice (circa 1760) of assuming for preliminary orbits that the chords to the middle radius vector were proportional to the time intervals. Gauss' work is doubtless indebted to the foundations which Lambert laid.

The function $Q(\phi) = \frac{3}{4}\left(\frac{\phi - \sin\phi}{\sin^3\frac{1}{2}\phi}\right)$ may be tabulated with the argument $x = \sin^2\frac{1}{2}\phi$ or expanded into a power series. We shall use the same method of determining the coefficients as we did for $X(x)$ on page 55, since this is the same function but with a slightly different argument. We have

$$\frac{dx}{d\phi} = \tfrac{1}{2}\sin\phi, \quad 1 - 2x = \cos\phi, \quad 2\sqrt{x(1-x)} = \sin\phi$$

Write

$$Q = \sum_0^\infty C_n x^n, \qquad \phi - \sin\phi - \frac{4}{3}Q(\phi)\sin^3\tfrac{1}{2}\phi = 0$$

Differentiate and substitute:

$$\frac{dQ}{dx} = \sum_1^\infty n C_n x^{n-1}; \quad \frac{2}{3}x\frac{dQ}{dx} + Q = (1-x)^{-1/2}$$

or

$$\frac{2}{3}\sum_1^\infty n C_n x^n + \sum_0^\infty C_n x^n = 1 + \tfrac{1}{2}x + \sum_2^\infty \frac{1\ 3\ 5\ \dots (2n-1)}{2^n n!}\, x^n$$

Thus

$$C_n = \frac{3}{2n+3}\,\frac{1\ 3\ 5\ \dots (2n-1)}{2^n n!}.$$

For $n = 0$, we have $C_0 = 1$; for $n = 1$, $C_1 = 3/10$; for $n = 2$, $C_2 = 9/56$; $c_3 = 5/48$; $C_4 = 105/1408$; etc.

Now $x = \sin^2\frac{1}{2}c = (r_i + r_j + S)/4a$ may be substituted into $Q(c)$ and $x = \sin^2\frac{1}{2}d = (r_i + r_j - S)/4a$ into $Q(d)$. Then the series expansion of ((5,45)) becomes

$$6k(t_j - t_i) = [(r_i + r_j + S)^{3/2} \mp (r_i + r_j - S)^{3/2}] +$$
$$\frac{3}{40a}[(r_i + r_j + S)^{5/2} \mp (r_i + r_j - S)^{5/2}] + \frac{9}{896a^2}[\ ] + \dots \qquad ((5,46))$$

The lower sign is for the case $\pi < 2f < 2\pi$, for then $-\pi < d < 0$.

In the case of a hyperbola, $1/a$ is replaced by $-1/a$, so that the signs of the odd powers are negative. In the case of a parabola, $1/a = 0$, and all the terms of ((5,46)) vanish except the first. This is known as Euler's equation, although it was first derived in geometrical terms by Newton. Some practical transformations have been introduced by Encke. Let

$$\frac{S}{r_i + r_j} = \sin\gamma, \qquad \sin\tfrac{1}{2}\gamma = \sqrt{2}\sin\tfrac{1}{3}\theta, \qquad \eta = \frac{2k(t_j - t_i)}{(r_i + r_j)^{3/2}}$$

Then $$\frac{6k(t_j - t_i)}{(r_i + r_j)^{3/2}} = (1 + \sin\gamma)^{3/2} - (1 - \sin\gamma)^{3/2} = (\cos\tfrac{1}{2}\gamma + \sin\tfrac{1}{2}\gamma)^3 - (\cos\tfrac{1}{2}\gamma - \sin\tfrac{1}{2}\gamma)^3 =$$
$$= 6\sin\tfrac{1}{2}\gamma - 4\sin^3\tfrac{1}{2}\gamma = 2^{3/2}[3\sin\tfrac{1}{3}\theta - 4\sin^3\tfrac{1}{3}\theta] = 2^{3/2}\sin\theta = 3\eta \qquad ((5,47))$$

where the proper trigonometric substitutions are readily perceived.

If the two terms $(1 \pm \sin\gamma)^{3/2}$ are expanded by the binomial theorem, we have

$$\eta = \sin\gamma - \frac{1}{4\,6}\sin^3\gamma - \frac{1}{4\,6\,8\,10}\sin^5\gamma - \ldots$$

and by inverting this series: $$\frac{S}{r_i + r_j} = \eta + \frac{1}{24}\eta^3 + \ldots = \eta\zeta \qquad ((5,48))$$

where $\eta\zeta$ may be tabulated with the argument η. This is given as Table 26 in the Bauschinger-Stracke Tafeln zur Theoretischen Astronomie, and a condensed table is given in the appendix.

This equation ((5,48)) specifies the condition which S must satisfy in a parabola if r_1, r_3, and T_2 are given. In this notation, $\kappa = (r_i + r_j)\cos\gamma$ and $l = (1 - \cos\gamma)/2\cos\gamma$, as the student may verify by comparing ((5,12)). Since the period of a parabola is infinite, the mean motion and the mean anomaly are zero, so that both g and x are also zero. Thus, in the case of a parabola

$$\bar{y} = \frac{1}{3}(1 + 2\sec\gamma) = \frac{1}{3}\left(1 + \frac{2}{\sqrt{1 - (\eta\zeta)^2}}\right) \qquad ((5,49))$$

In the appendix, a condensed table of $\bar{y}$ is given with the argument η.

We are now prepared to tackle the problem of determining a preliminary parabolic orbit. The following derivation of Olbers' method is patterned after the one given by Bengt Stroemgren in Kobenhavn Obs. Publ. No. 66, although the principal equation in the solution has been modified. The fundamental equation is

$$c_1\mathbf{p}_1 - \mathbf{p}_2 + c_3\mathbf{p}_3 = c_1\mathbf{R}_1 - \mathbf{R}_2 + c_3\mathbf{R}_3 = \mathbf{V} \qquad ((5,50))$$

Operate upon both sides of this equation by $\cdot(\mathbf{p}_2^* \times \mathbf{U})$, where $\mathbf{U}$ is coplanar with $\mathbf{V}$ and $\mathbf{p}_2$:

$$c_1\rho_1(\mathbf{p}_1^* \cdot \mathbf{p}_2^* \times \mathbf{U}) + c_3\rho_3(\mathbf{p}_3^* \cdot \mathbf{p}_2^* \times \mathbf{U}) = \mathbf{V} \cdot \mathbf{p}_2^* \times \mathbf{U} = 0$$

Then $\rho_3 = M\rho_1$, where $$M = -\frac{c_1(\mathbf{p}_1^* \cdot \mathbf{p}_2^* \times \mathbf{U})}{c_3(\mathbf{p}_3^* \cdot \mathbf{p}_2^* \times \mathbf{U})}. \qquad ((5,51))$$

This M, along with Euler's equation, is the core of Olbers' method. This expression for M becomes an indeterminate form if the object is too close to the opposition point on the sky. We have already determined the dynamical condition which the chord must satisfy if the motion is to be in a parabola. The geometrical expression for the chord is easily found. Each of these conditions is expressed in terms of two radius vectors, but by means of M they may both be made to depend only upon ρ_1. By equating the two expressions for the chord, we have one equation for the determination of ρ_1.

Write $\mathbf{R}_2 = C_1\mathbf{R}_1 + C_3\mathbf{R}_3$, where the C's are the triangle ratios for the motion of the Earth.

Then $$\mathbf{V} = (c_1 - C_1)\mathbf{R}_1 + (c_3 - C_3)\mathbf{R}_3$$
$$= \frac{T_2^2}{6}\left(\frac{1}{r_2^3} - \frac{1}{R_2^3}\right)\left[c_1^0(1 - c_1^{0\,2})\mathbf{R}_1 + \ldots + c_3^0(1 - c_3^{0\,2})\mathbf{R}_3 + \ldots\right] = X\mathbf{R}_2 + \ldots$$

Therefore, for the first approximation we shall use $\mathbf{U} = \mathbf{R}_2$ and $\frac{c_1}{c_3} = \frac{T_1}{T_3}$ in ((5,51)), and assume that $\mathbf{U} \times \mathbf{V} = 0$. The geometrical expression for the chord is given by

$$S^2 = (\mathbf{r}_3 - \mathbf{r}_1)\cdot(\mathbf{r}_3 - \mathbf{r}_1)$$

where we now use $\mathbf{r}_3 = \rho_3\mathbf{p}_3^* - \mathbf{R}_3 = M\rho_1\mathbf{p}_3^* - \mathbf{R}_3$, and therefore

$$\mathbf{r}_3 - \mathbf{r}_1 = (M\mathbf{p}_3^* - \mathbf{p}_1^*)\rho_1 - (\mathbf{R}_3 - \mathbf{R}_1).$$

Then $$[S(g)]^2 = (M\mathbf{p}_3^* - \mathbf{p}_1^*)\cdot(M\mathbf{p}_3^* - \mathbf{p}_1^*)\rho_1^2 - 2(M\mathbf{p}_3^* - \mathbf{p}_1^*)\cdot(\mathbf{R}_3 - \mathbf{R}_1)\rho_1 + (\mathbf{R}_3 - \mathbf{R}_1)\cdot(\mathbf{R}_3 - \mathbf{R}_1)$$
$$= A + B\rho_1 + C\rho_1^2$$

Also $$r_1^2 = R_1^2 - 2(\mathbf{R}_1\cdot\mathbf{p}_1^*)\rho_1 + \rho_1^2 = a + b\rho_1 + c\rho_1^2$$

and $$r_3^2 = R_3^2 - 2M(\mathbf{R}_3\cdot\mathbf{p}_3^*)\rho_1 + M^2\rho_1^2 = \alpha + \beta\rho_1 + \gamma\rho_1^2$$

The dynamical value of the chord must satisfy the equation

$$S(d) = (\eta\zeta)(r_1 + r_3)$$

Therefore if we write

$$\Delta(\rho_1) = S(g) - S(d) = 0 \qquad ((5,52))$$

we shall have one equation for the determination of ρ_1.

Stroemgren has reduced the preliminary solution to an ingenious nomogram, but this involves the computation of extra auxiliary quantities and his paper may not be readily available for the reader to use, so that all these details have not been presented. We shall solve ((5,52)) simply by inverse interpolation between the values of $\Delta(\rho_1)$ which result from assumed values of ρ_1.

The value of ρ_1 which causes $\Delta(\rho_1)$ to vanish is not necessarily the final solution on account of the approximations we have made in computing M. Let us consider the situation in the following manner. Any temporarily adopted value of ρ_1 fixes $\mathbf{r}_1$ and $\mathbf{r}_3$, and by successive approximations we may determine $\mathbf{r}_2$. First assume a value of r_2, based on the values of r_1 and r_3. This permits the determination of the η's, y's, and the c's, so that a better approximation to r_2 may be found. Repeat the solution with the new value of r_2 until it is final. Then the final value of $\mathbf{p}_2$ will produce residuals, $\Delta\alpha$ and $\Delta\delta$ when compared with the middle observation. Of course, if the motion of the object is not in a parabolic orbit it will be impossible to remove the residuals. But if the residuals are due to the error in the adopted value of M, we may proceed to correct this in any one of several ways. One method would be to determine one or two other pairs of residuals by repeating the solution with M + 0.1 and M - 0.1; and then interpolate for the best result. Another is to recompute M by means of the known values of the c's, using $\mathbf{U} = c_1\mathbf{R}_1 - \mathbf{R}_2 + c_3\mathbf{R}_3$. But Stroemgren adopts still another method, the method of "false position". If $\mathbf{U}$ is held fixed, then M may be considered to be a function of $\mathbf{p}_2^*$. Since $\mathbf{p}_2^*$(observed) produces a solution which yields the middle position at $\mathbf{p}_2^*$(computed), then if we use a fictitious $\mathbf{p}_2^* = 2\,\mathbf{p}_2^*$(observed) - $\mathbf{p}_2^*$(computed) in M, we may expect to get a solution which yields the middle position at $\mathbf{p}_2^*$(observed). Therefore M is recomputed exactly as before, except for the use of this fictitious $\mathbf{p}_2^*$, (i.e. the "false position") and then the solution is repeated.

Once the final value of ρ_1 is determined, then $\mathbf{r}_1$ and $\mathbf{r}_3$ are the constants of integration and we are at a corresponding position with the Gaussian method of solution, except that e = 1. Then

$$q = \frac{|\mathbf{r}_1 \times \mathbf{r}_3|^2 y_2^2}{2\,T_2^2} \qquad \text{and} \qquad \tan^2\tfrac{1}{2}v = \frac{r - q}{q} \qquad ((5,53))$$

The same principles which underlie Olber's method may be used to condition the general solution of the Gaussian method if the period or the semi-major axis is to be adopted in advance. Compute M and S(g) in the same way as before. Then $x(c) = (r_i + r_j + S)/4a$ and $x(d) = (r_i + r_j - S)/4a$ are used as arguments for Q(c) and Q(d), resp. and finally from ((5,45))

$$\Delta(\rho_1) = 6k(t_j - t_i) - Q(c)(r_i + r_j + S)^{3/2} + Q(d)(r_i + r_j - S)^{3/2} = 0 \qquad ((5,54))$$

In other respects the solution is the same as for the parabola. The test of the validity of such a conditioned solution is always the size of the outstanding residual which cannot be removed from the middle observation; the first and third observations are always exactly represented. This procedure is of practical value in the case that a new orbit is to be computed from a few observations extending over a short arc for a known or suspected object which has just been rediscovered. Since the same initial data are thus expected to yield only five unknowns instead of six, the solution will be more determinate.

We shall now illustrate each of these methods with an example. The following observations are of Comet Oterma II = 1942-f. In accordance with the accepted practice, the positions and elements of all newly discovered objects are referred to the mean equator and equinox for the beginning of the year. First, we shall suppose that only the first three observations are available; and we shall solve for a parabolic orbit. The numerical values in ( ) are obtained after the values of ρ_1 are determined from the solution.

Observations of Comet Oterma II = 1942-f

1942 UT		α (1942.0)	δ (1942.0)	
Nov. 11.18242	= JD 2430674.68242	$4^h\ 10^m\ 21^s.29$	$+2°\ 00'\ 04''.8$	Yerkes
12.24299	675.74299	4 10 12.16	+2 19 48.5	Lick
13.12670	676.62670	4 10 03.50	+2 36 45.3	Yerkes
27.06738	690.56738	4 06 03.32	+8 04 54.9	Yerkes
Dec. 14.10274	707.60274	4 01 05.74	+16 39 23.6	Yerkes

	-0.6605904	-0.6465855	-0.6347413	-0.4299053	-0.1461377
R_i	-0.6765258	-0.6875155	-0.6964900	-0.8147839	-0.8930602
	-0.2934346	-0.2982001	-0.3020980	-0.3534086	-0.3873496
	+0.4600940	+0.4605830	+0.4610431	+0.4722098	+0.4750480
p_i^*	+0.8871832	+0.8866851	+0.8862062	+0.8702028	+0.8319676
	+0.0349227	+0.0406574	+0.0455823	+0.1405887	+0.2866341

i	1	2	3
t_i	674.68242	675.74299	676.62670
	(.67792)	(.73853)	(.62227)
$2\,T_i$	0.0304034 (45)	0.0668916 (40)	0.0364882 (95)
m, T_2^2, n	0.4545179	0.00111862	0.5454821
	$p_2^* \times R_2$	$M p_3^* - p_1^*$	$R_3 - R_1$
	-0.2364570	-0.0065659	+0.0258491
	+0.1110574	-0.0154222	-0.0199642
	+0.2566589	+0.0099166	-0.0086634

$$M = \frac{0.4545179}{0.5454821}\,\frac{0.001301}{0.001102} = 0.9837$$

a, b, c	$r_1^2 =$ +0.9801707	+ 1.8287670 ρ_1	+ 1.0	ρ_1^2
α, β, γ	$r_3^2 =$ +0.9792580	+ 1.8171835	+ 0.9676657	
A, B, C	$S(g)^2 =$ +0.0011418	- 0.0001045	+ 0.0003793	

ρ_1	1.0	0.9	0.8	0.7
ρ_1^2	1.0	0.81	0.64	0.49
r_1	1.9516500	1.8536615	1.7558999	1.6584051
r_3	1.9401307	1.8435109	1.7470864	1.6508915
$r_1 + r_3$	3.8917807	3.6971724	3.5029863	3.3092966
η	0.0087126	0.0094095	0.0102027	0.0111114
$\eta\zeta$	0.0087126	0.0094095	0.0102027	0.0111114
S(d)	0.0339075	0.0347885	0.0357399	0.0367709
S(g)	0.0376377	0.0368101	0.0360687	0.0354190
Δ	+0.003730	+0.002022	+0.000329	-0.001352

0.7	-1352			
		+1681		
0.8	+ 329		+12,	$0 = +329 + 1687n + 6n^2$, $n = -0.1952$
		+1693		
0.9	+2022			

ρ_i	0.78048		0.76776	$r_1 \times r_3$	R
	+1.0196846	+1.0028163	+0.9887118	+0.0199124	+0.329322
r_i	+1.3689545	+1.3733164	+1.3768837	-0.0266588	-0.072265
	+0.3206911	+0,3296476	+0.3370943	+0.0504856	+0.941448
				0.0604648	
r^2	3.0166359	3.0003060	2.9869923		
r	1.7368465	1.73218	1.7282917	q 1.63415	171.73965
		1.7321391		$\sqrt{q}$ 1.27834	$k/\sqrt{2}\,q^{3/2}$ .00582277
$r_i + r_j$	3.4604717	3.4651382	3.4690265		
η	0.0047232	0.0103703	0.0056475	T = JD 2430718.6327	
y	1.0000075	1.0000359	1.0000106		

c	0.4545298	P_2	0.5454969	
		+0.3562308		
		+0.6858009		
		+0.0314475		
		0.7734411		
	cot α	+0.5194376		
	sin δ	+0.0406592		
	α	4 10 12.23	(-0.07)	
	δ	+2 19 48.9	(-0.4)	

cot Ω	+0.219434	Ω	77°.6235
sin i	0.337158	i	19.7038
r sin i	0.585592		
cos u	+0.903892		
sin u	-0.427760	u	-25.3255
$\tan^2\frac{1}{2}v$	0.062844	v	-28.1466
$\tan\frac{1}{2}v$	-0.250687	ω	2.8211

This is an entirely satisfactory solution, although from an arc of only two days it is to be expected that there is a wide range of sets of elements, any of which would represent the observations equally well. Some indication of the determinateness of the solution is given by the magnitude of $d\Delta/d\rho$ or the coefficient of n in the solution for ρ; in this case it is 0.01687, which is fairly large for such a short arc. However, according to Whipple, a parabolic solution from an arc of eight days leaves a residual of 30″ in the first position (cf. Harvard Announcement Card 640).

An ephemeris for an object travelling on a parabolic orbit is most readily derived by using the equation

$$\mathbf{r} = q\mathbf{P}(1 - \tan^2\tfrac{1}{2}v) + 2q\mathbf{Q}\tan\tfrac{1}{2}v \qquad ((5,55))$$

where $q\mathbf{P}$ and $2q\mathbf{Q}$ represent the vectorial constants for the equator and $\tan\frac{1}{2}v$ is obtained from the solution of ((3,29)), either by means of ((3,30)) or the tables in the Kobenhavn Obs. Publ. No. 58. The student may now compute the ephemeris as an exercise and to see how well the predictions based upon this solution compare with later observations, especially the fourth and fifth above.

An examination of Galle's Cometenbahnen shows that the orbit of this comet is situated very similar to that of Comet 1867 I. The effects of precession on the elements over the intervening years are less than the lack of exact agreement between the two sets of elements, but if it were desired to take them into account it is most readily accomplished by means of Table II, Planetary Coordinates. For comparison, approximate values of the elements in the Cometenbahnen are as follows: i = 18°.21, Ω = 78°.46, ω = 357°.52, q = 1.577, e = 0.865, T = 1867.05. The period is given as $40^y \pm 2$, so that if these two comets are identically the same object there must have been two revolutions during the intervening time. This gives P = 37ʸ.97 or (with sufficient accuracy) we may adopt a = 11.30.

To illustrate the conditioned solution, we shall now use the first, fourth, and fifth of the above observations and adopt 11.3 as the value for the semi-major axis. From our preliminary parabolic solution we may compute values of the c's corresponding to the times of these three observations, and thus obtain a first approximation to U in order to find M by means of ((5,51)). Then from the solution of Lambert's equation we shall obtain more accurate values of the c's and the residuals corresponding to this value of M. It is apparent that in this method of solution, after the first and third positions are adopted, then the residuals of the middle place are functions of the arbitrary parameter M. Therefore another value of M is adopted, etc. until the residuals can be reduced to a minimum. That is then the best solution that can be obtained by this method with the adopted value of the semi-major axis.

JD	2430674.68242	690.56738	707.60274
$2T_i$	0.5860896	1.1326006	0.5465109
m, $T_2^2/12$, n	0.5174725	0.0267247	0.4825275

From the parabolic solution:

N		-0.163444	-0.064251
$\tan\frac{1}{2}v$		-0.155396	-0.062931
r	1.736846	1.673611	1.640622
$r_i + r_j$	3.314233	3.377468	3.410457
η	0.0971381	0.1824693	0.0867720
y	1.0031702	1.0114162	1.0025257
c	0.5217261		0.4868066

V	$p_2^* \times V$	$M p_3^* - p_1^*$	$R_3 - R_1$
+0.0141173	+0.0064200	-0.0325508	+0.5144527
+0.0270751	-0.0035646	-0.1384124	-0.2165344
+0.0117518	+0.0005002	+0.2230480	-0.0939150

Because it agrees very closely with the computed value, we adopt the following exact value for the parameter M.

$$M = 0.9$$

$$\begin{aligned}
r_1^2 &= +0.9801707 + 1.8287670\,\rho_1 + 1.0\quad \rho_1^2 \\
r_3^2 &= +0.9689525 + 1.6622049 \quad + 0.81 \\
S(g)^2 &= +0.3203688 + 0.0154447 \quad + 0.0699680 \\
6k(t_3 - t_1) &= +3.397802 \quad + 0.00005956
\end{aligned}$$

Notice that the "light time" correction is simply $-0.0005956(M - 1)\,\rho_1$.

Solution of Lambert's equation

ρ_1	0.78	0.70	0.7055	0.70558
ρ_1^2	0.6084	0.4900	0.4977302	0.4978431
r_1	1.7363781	1.6584051	1.6637596	1.6638375
r_3	1.6608059	1.5904075	1.5952428	1.5953131
S(g)	0.6123595	0.6045365	0.6050539	0.6050614
(+)	4.0095435	3.8533491	3.8640563	3.8642120
$(+)^{3/2}$	8.0286478	7.5641046	7.5956539	7.5961130
(-)	2.7848245	2.6442761	2.6539485	2.6540892
$(-)^{3/2}$	4.6472578	4.2999182	4.3235326	4.3238765
x(c)	0.0887067	0.0852511	0.0854880	0.0854914
Q(c)	1.0279543	1.0268121	1.0268903	1.0268914
x(d)	0.0616112	0.0585017	0.0587157	0.0587188
Q(d)	1.0191189	1.0181223	1.0181908	1.0181918
Δ	-0.119126	+0.008772	+0.000122	-0.0000035

The first value, 0.78, is chosen as a first approximation from the previous parabolic solution. Since the residual is negative, the geometrical chord is too large, and so the distance must be diminished. The next value, 0.70, is simply taken as a smaller even number; and therefater the successive corrections are based on the numerically estimated rate of change of $d\Delta/d\rho_1$. In the appendix a condensed table of $Q(\phi)$ is given.

By successive approximations we shall now determine the residuals of the middle place corresponding to the two adopted outer radius vectors. For the c's we again use the Gibbs formulas with Koziel's correction term. It is apparent from the small value of C that these formulas are sufficiently accurate for the present arc.

ρ	0.7055778		0.6350200
	+0.9852225	+0.7326954	+0.4478027
r	+1.3025026	+1.3727675	+1.4213763
	+0.3180753	+0.4435342	+0.5693680
		2.625	
r^2	2.7683483	2.6180557	2.5450178
r^{-3}	0.2171046	0.2351289	0.2462996
		0.2360650	
B_i	+0.0057391	+0.0333977	+0.0076069
mn, C, Div.	0.2496947	+0.0000079	+0.9921472
		+0.0000095	+0.9921160

0.5222161		0.4872599
0.5222322		0.4872756
	r_2	P_2
	+0.7327182	+0.3028129
	+1.3728108	+0.5580269
	+0.4435483	+0.0901403
	2.6182205	0.6412604
	0.2360428	(+0.07, +4.4)

Now the corrections for "light time" may be applied.

JD	674.67835	690.56368	707.59908
m, $T_2^2/12$, n	0.5174673	0.0267253	0.4825327
B_1	+0.0057395	+0.0333985	+0.0076068
mn, C, Div.	0.2496949	+0.0000094	+0.9921165

c	0.5222267	$\mathbf{p}_2$	0.4872806
		+0.3028098	
		+0.5580268	
		+0.0901408	
		0.6412590	
		(+0.01, +4.2)	

With these c's we recompute **V** and M.

$\mathbf{V}$	$\mathbf{p}_2^* \times \mathbf{V}$	$M\mathbf{p}_3^* - \mathbf{p}_1^*$	$\mathbf{R}_3 - \mathbf{R}_1$
+0.0137173	+0.0062395	-0.0320758	+0.5144527
+0.0263132	-0.0034648	-0.1375804	-0.2165344
+0.0114213	+0.0004885	+0.2233346	-0.0939150

This time we adopt the value M = 0.9005

$$
\begin{aligned}
r_1^2 &= 0.9801707 + 1.8287670\,\rho_1 + 1.0 \quad \rho_1^2\\
r_3^2 &= 0.9689525 + 1.6631284 \quad + 0.8109002\\
S(g)^2 &= 0.3203688 + 0.0154074 \quad + 0.0699015\\
6k(t_3 - t_1) &= 3.397802 \quad + 0.00005926
\end{aligned}
$$

ρ_1	0.7055778	0.7056288
ρ_1^2	0.4978400	0.4979120
r_1	1.6638354	1.6638850
r_3	1.5956558	1.5957007
S(g)	0.6050121	0.6050169
(+)	3.8645033	3.8646026
$(+)^{3/2}$	7.5969720	7.5972648
(-)	2.6544791	2.6545688
$(-)^{3/2}$	4.3248293	4.3250485
x(c)	0.0854979	0.0855001
Q(c)	1.0268935	1.0268943
x(d)	0.0587274	0.0587294
Q(d)	1.0181945	1.0181952
Δ	+0.000080	0.000000

ρ	0.7056288		0.6354187
	+0.9852460	+0.7328170	+0.4479921
$\mathbf{r}$	+1.3025478	+1.3729906	+1.4217080
	+0.3180771	+0.4436042	+0.5694823
r^2	2.7685135	2.6189082	2.5462606
r^{-3}	0.2170851	0.2359498	0.2461193
C, Div.		+0.0000094	0.9921196

c	0.5222250	$\mathbf{p}_2$	0.4872784
		+0.3029117	
		+0.5582067	
		+0.0901956	
		0.6414713	
		(+0.09, -3.9)	

From these two solutions we now interpolate the following:

$$M = 0.90025, \quad \rho_1 = 0.7056033, \quad \rho_3 = 0.6352194, \quad (+0.05, +0.1).$$

These values enable us to determine $\mathbf{r}_1$ and $\mathbf{r}_3$, and then the elements and ephemeris may be computed as on pages 63 to 65.

When the conditions exhibited in equation ((3,46)) make it imperative that four observations must be used to obtain the preliminary orbit, it is then necessary to write down one equation like the first of ((5,36)), but relating the 1st, 2nd, and 4th observations, and another similar equation relating the 1st, 3rd, and 4th observations. It is then these two equations which are solved for the coordinates at the times of the 1st and 4th observations. This solution is not indeterminate, even though ((5,37)) would be. The only disadvantage of this method is that one is obliged to deal with two pairs of triangle-ratios instead of one pair.

CHAPTER 6

IMPROVEMENT OF THE ORBIT

Εἰ πρῶτον μὴ εὐτυχήσω, αὖθις αὖ πάλιν πείρασαι.

In the two preceding chapters, we have simulated the situation in which there are available only a few observations of an unidentified object extending over a short arc, and we have derived preliminary elements by very simple methods. In this chapter we shall simulate two other situations: one in which the observations extend over a longer arc of several months, and another in which we wish to obtain the best results from observations extending over a number of years. The whole problem now takes on an entirely different aspect and depends upon a different basic principle.

The student should be careful to understand this distinction clearly. We are no longer concerned directly with dynamical conditions as such, although we can not, of course, divorce the problem from its intrinsic dependence upon the dynamical conditions. But the formal operations of the Calculus, differentiation, etc., are no longer applied to time-rates-of-change and dynamical conditions of acceleration; they are now applied to geometrical, differential effects which are due to the differential changes that we permit the previously known preliminary elements to take. These are first order differential effects.

The fundamental concept is now the equation for the total differential of a function of several independent variables:

$$dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \dots + \frac{\partial F}{\partial x_n} dx_n \qquad ((6,1))$$

or in Cracovian notation

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ \vdots \\ \vdots \end{Bmatrix} \begin{Bmatrix} \partial F/\partial x_1 \\ \partial F/\partial x_2 \\ \vdots \\ \vdots \end{Bmatrix} = \{dF\} \qquad ((6,2))$$

The function F is either of the two independent angular coordinates, α or δ, which define the position of the object on the sky. The values of these coordinates that are computed from the preliminary elements do not agree exactly with the observations; they leave the residuals (in the sense "observed" minus "computed") $\cos\delta\,\Delta\alpha$ (O - C) and $\Delta\delta$ (O - C). We wish to solve for such numerical values of the differentials of the independent variables in ((6,1)) that the total differential of F will be changed by the amount of the (O - C) residual; in other words, the value of F that is computed from the corrected elements should agree with the observed value. Since several transformations of variables are involved, it is simplest to determine the final equations of condition in several steps. At the instant of any observation, we have $\mathbf{p} = \mathbf{r} + \mathbf{R}$ and $d\mathbf{p} = d\mathbf{r}$. The scalar components of this differential relationship are

$$\begin{aligned} -\rho\cos\delta\sin\alpha\,d\alpha - \rho\sin\delta\cos\alpha\,d\delta + \cos\delta\cos\alpha\,d\rho &= dx \\ +\rho\cos\delta\cos\alpha\,d\alpha - \rho\sin\delta\sin\alpha\,d\delta + \cos\delta\sin\alpha\,d\rho &= dy \\ +\rho\cos\delta\,d\delta + \sin\delta\,d\rho &= dz \end{aligned} \qquad ((6,3))$$

From these equations we may solve for our total differentials in terms of x, y, z at the time of the observation as independent variables.

$$\begin{aligned} -\sin\alpha\,dx + \cos\alpha\,dy &= \rho\cos\delta\,d\alpha \\ -\sin\delta\cos\alpha\,dx - \sin\delta\sin\alpha\,dy + \cos\delta\,dz &= \rho\,d\delta \end{aligned}$$

or

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} \begin{Bmatrix} -\sin\alpha/\rho & -\sin\delta\cos\alpha/\rho \\ +\cos\alpha/\rho & -\sin\delta\sin\alpha/\rho \\ 0 & +\cos\delta/\rho \end{Bmatrix} = \begin{Bmatrix} \cos\delta\, d\alpha \\ d\delta \end{Bmatrix} \qquad ((6,4))$$

These two equations ((6,4)) are applicable at any time, and they simply express the effect of a differential change in the rectangular coordinates upon the spherical coordinates.

The partial differential coefficients in the middle Cracovian may also be obtained directly, e.g. since

$$\alpha = \arctan\frac{y+Y}{x+X}, \quad \cos\delta\frac{\partial\alpha}{\partial x} = \frac{-(y+Y)\cos\delta}{(x+X)^2+(y+Y)^2} = -\sin\alpha/\rho\,.$$

As an exercise, the student may check the other terms in this way. It is to be noted that the two columns of elements in the middle Cracovian of ((6,4)) are the components of vectors which are mutually orthogonal with $\mathbf{p}$ and oriented with respect to the equatorial coordinate system, i.e. they are the vectors along which differential displacements will produce differential changes in α and δ, respectively. In effect, this Cracovian is an operator which perforce effects a transformation to the special set of reference axes along which our preassigned, total differential corrections have been measured, irrespective of where upon the sky the observation is situated. Since we can make no direct measurement of ρ along the line of sight, there is no equation of condition for a displacement along $\mathbf{p}$, the third axis of this special set.

It is not feasible to deal with all these different sets of independent variables (the x, y, z at the time of each observation), and so we must eliminate all of them in terms of the one set of six variables to which the final correction can be determined and applied, in other words, the six constants of integration or elements of the orbit. In the first case, we shall adopt as our set of elements the components of $\mathbf{r}_0$ and $\mathbf{r}'_0$ at some convenient epoch, t_0. From the differentials of ((4,4)) we have

$$d\mathbf{r} = f\,d\mathbf{r}_0 + g\,d\mathbf{r}'_0 + \mathbf{r}_0 df + \mathbf{r}'_0 dg \qquad ((6,5))$$

To find expressions for df and dg, let us limit ourselves to three terms in each of the series of ((4,3)), and take differentials with respect to the elements which are to be varied, x_0, y_0, z_0, x'_0, y'_0, z'_0. The student will observe that there is a distinct difference between this and the differentiations which were performed in deriving the f and g series. In that case we were interested in determining the effect of a change in the time upon the object's position in its orbit while the elements remained fixed, but now we wish to determine the effect of a change in the elements while the time remains fixed. In the former case we had one independent variable, now we have six. To simplify the notation, let w represent a summation over x, y, and z, and omit the zero subscript with the understanding that the following applies at the time t_0. Following ((4,4)), we have

$$df = -\frac{1}{2}\tau^2(1-\sigma\tau)\,d\mu + \frac{1}{2}\tau^3\mu\,d\sigma, \qquad dg = -\frac{1}{6}\tau^3(1-\frac{3}{2}\sigma\tau)\,d\mu + \frac{1}{4}\tau^4\mu\,d\sigma.$$

$$d\mu = -\frac{3\,dr}{r^4} = -3\frac{\mu}{r^2}(w\,dw), \qquad d\sigma = \frac{w\,dw'}{r^2} + \frac{w'dw}{r^2} - 2\frac{\sigma}{r^2}(w\,dw)$$

and $d\mu$ and $d\sigma$ are to be substituted into df and dg.

The coefficients of the unknowns, dx_0, etc., may be expressed in Cracovians as follows:

$$\begin{array}{cccccc} dx_0 & dy_0 & dz_0 & dx'_0 & dy'_0 & dz'_0 \\ \end{array}$$
$$\begin{Bmatrix} x_0 & y_0 & z_0 & 0 & 0 & 0 \\ x'_0 & y'_0 & z'_0 & x_0 & y_0 & z_0 \end{Bmatrix} \begin{Bmatrix} \tau^2\mu(3-5\sigma\tau)/2r^2, & \tau^3\mu(2-5\sigma\tau)/4r^2 \\ \tau^3\mu/2r^2, & \tau^4\mu/4r^2 \end{Bmatrix} = \begin{Bmatrix} df \\ dg \end{Bmatrix}$$

These expressions are to be substituted into the last two terms of ((6,5)). Similarly ((6,5)) may be expressed in the notation of Cracovians as follows:

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} f & 0 & 0 & g & 0 & 0 \\ 0 & f & 0 & 0 & g & 0 \\ 0 & 0 & f & 0 & 0 & g \end{Bmatrix} + \begin{Bmatrix} x_0 df \\ y_0 df \\ z_0 df \end{Bmatrix} + \begin{Bmatrix} x'_0 dg \\ y'_0 dg \\ z'_0 dg \end{Bmatrix} \qquad ((6,6))$$

This is to be substituted for the left member of ((6,4)). But it is possible to combine all of this into one compounded Cracovian in the following way. Let

$$\left\{\begin{matrix} -\sin\alpha/\rho & -\sin\delta\cos\alpha/\rho \\ +\cos\alpha/\rho & -\sin\delta\sin\alpha/\rho \\ 0 & +\cos\delta/\rho \end{matrix}\right\}\left\{\begin{matrix} x_0 & x_0' \\ y_0 & y_0' \\ z_0 & z_0' \end{matrix}\right\} = \left\{\begin{matrix} (5) & (7) \\ (6) & (8) \end{matrix}\right\} \qquad ((6,7))$$

Then

$$\begin{array}{cccccc} dx_0 & dy_0 & dz_0 & dx_0' & dy_0' & dz_0' \end{array}$$
$$\left\{\begin{matrix} f & 0 & 0 & g & 0 & 0 \\ 0 & f & 0 & 0 & g & 0 \\ 0 & 0 & f & 0 & 0 & g \\ \cdots & \cdots & df & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & dg & \cdots & \cdots \end{matrix}\right\}\left\{\begin{matrix} -\sin\alpha/\rho & -\sin\delta\cos\alpha/\rho \\ +\cos\alpha/\rho & -\sin\delta\sin\alpha/\rho \\ 0 & +\cos\delta/\rho \\ (5) & (7) \\ (6) & (8) \end{matrix}\right\} = \left\{\begin{matrix} \cos\delta\, d\alpha \\ d\delta \end{matrix}\right\} \qquad ((6,8))$$

This arrangement obliges us to multiply by df and dg only once instead of three times.

The residuals to be removed may be computed in any one of several ways.

$$\rho\cos\delta\, d\alpha = \rho\cos\delta\sin\Delta\alpha(O - C) = (x + X)\sin\alpha - (y + Y)\cos\alpha$$
$$\rho\, d\delta = \rho\sin\Delta\delta(O - C) = [(x + X)^2 + (y + Y)^2]^{1/2}\sin\delta - (z + Z)\cos\delta$$

where α and δ are the observed values and the geocentric rectangular coordinates are the computed values. These formulas give the residuals in radians. Alternatively, the residuals may be derived by a direct evaluation of the computed position and subtraction of this from the observed position. Then the corrections must be converted to radians before they are applied.

The equations ((6,8)) have been developed in a completely general manner and their use will not be restricted by the convergence of the f and g series if we can obtain closed expressions for df and dg. The following development is based upon Bower's presentation in the Lick Observatory Bulletin 445. Let $\Delta E = E - E_0$, $F = a(1 - \cos\Delta E)$, $G = a^{1/2}\sin\Delta E$. From ((4,22))

$$f = \frac{-e\cos E_0 + \cos E\cos E_0 + \sin E\sin E_0}{1 - e\cos E_0} = 1 - \frac{a(1 - \cos\Delta E)}{r_0} = 1 - \frac{F}{r_0}$$

$$g = [(\cos E_0 - e)\sin E - (\cos E - e)\sin E_0]\, a^{3/2}$$
$$= [\sin\Delta E + (M - E) - (M_0 - E_0)]\, a^{3/2} = \tau - [\Delta E - \sin\Delta E]\, a^{3/2}$$

Also
$$g = [\sin\Delta E - e\sin E + e\sin E_0]\, a^{3/2}$$
$$= [\sin\Delta E + e\sin E_0 - e\sin E_0\cos\Delta E - e\cos E_0\sin\Delta E]\, a^{3/2}$$
$$= r_0 a^{1/2}\sin\Delta E + a(1 - \cos\Delta E)(r r')_0 = G r_0 + F(r r')_0$$

Now let us take differentials with respect to the elements on both sides of all three of these expressions.

$$df = \frac{a}{r_0^2}(1 - \cos\Delta E)\, dr_0 - \frac{(1 - \cos\Delta E)}{r_0}\, da - \frac{a}{r_0}\sin\Delta E\, d\Delta E$$

$$dg = -\frac{3}{2}a^{1/2}(\Delta E - \sin\Delta E)\, da - a^{3/2}(1 - \cos\Delta E)\, d\Delta E$$

$$dg = a^{1/2}\sin\Delta E\, dr_0 + [\tfrac{1}{2}r_0 a^{-1/2}\sin\Delta E + (1 - \cos\Delta E)(r r')_0]\, da$$
$$+ a(1 - \cos\Delta E)\, d(r r')_0 + [r_0 a^{1/2}\cos\Delta E + a(r r')_0\sin\Delta E]\, d\Delta E$$

Thus
$$df = \frac{F}{r_0^2}\, dr_0 - \frac{F}{a r_0}\, da - \frac{G a^{1/2}}{r_0}\, d\Delta E$$
$$dg = -\frac{3}{2}\frac{(\tau - g)}{a}\, da - F a^{1/2}\, d\Delta E$$
$$dg = G\, dr_0 + \left[\frac{G r_0}{2a} + \frac{F(r r')_0}{a}\right] da + F\, d(r r')_0 + \left[r_0 - \frac{F r_0}{a} + G(r r')_0\right] a^{1/2}\, d\Delta E$$

Equate these last two expressions in order to solve for $d\Delta E$. The coefficient of $d\Delta E$ reduces to $r\, a^{1/2}$ by means of the substitution

$$r = a(1 - e\cos E) = a - ae\cos E_0 \cos\Delta E + ae\sin E_0 \sin\Delta E$$

$$= a - (1 - r_0/a)(a - F) + (r r')_0 G = F + r_0 - F r_0/a + G(r r')_0$$

Also let

$$L = \frac{3\tau - g - G r_0}{r} = \frac{3(\tau - g) + 2F(r r')_0 + G r_0}{r}$$

Thus

$$a^{1/2}\, d\Delta E = -\frac{G}{r} dr_0 - \frac{L}{2}\frac{da}{a} - \frac{F}{r} d(r r')_0$$

Since

$$\frac{1}{a} = \frac{2}{r_0} - V_0^2, \qquad \frac{da}{a^2} = \frac{2}{r_0^2} dr_0 + d(V_0^2)$$

We may now eliminate all of the above variables and obtain our original expressions, df and dg, in terms of the set we are seeking to correct, if we substitute

$$r_0 dr_0 = w_0 dw_0, \quad \tfrac{1}{2} d(V_0^2) = w_0' dw_0', \quad d(r r')_0 = w_0 dw_0' + w_0' dw_0$$

The algebraic reductions are left as an exercise for the student. The results are

$$df = (1)\, r_0 dr_0 + \tfrac{1}{2}(2M)\, d(V_0^2) + (3)\, d(r r')_0$$

$$dg = (2)\, r_0 dr_0 + \tfrac{1}{2}(2N)\, d(V_0^2) + (4)\, d(r r')_0$$

where

$$(2M) = \frac{a}{r_0}(GL - 2F) = \frac{GL - 2F}{1 - e\cos E_0} \qquad (4) = F^2/r$$

$$(2N) = (3g - 3\tau + FL)a \qquad (3) = \frac{FG}{r r_0} \qquad ((6,9))$$

$$(1) = \left[\frac{G^2 r_0}{r} + (2M) + F\right]\frac{1}{r_0^3} \qquad (2) = \frac{(2N)}{r_0^3} + (3)$$

and finally

$$\begin{array}{cccccc} dx_0 & dy_0 & dz_0 & dx_0' & dy_0' & dz_0' \end{array}$$
$$\begin{Bmatrix} x_0 & y_0 & z_0 & 0 & 0 & 0 \\ x_0' & y_0' & z_0' & x_0 & y_0 & z_0 \\ 0 & 0 & 0 & x_0' & y_0' & z_0' \end{Bmatrix} \begin{Bmatrix} (1) & (2) \\ (3) & (4) \\ (2M) & (2N) \end{Bmatrix} = \begin{Bmatrix} df \\ dg \end{Bmatrix} \qquad ((6,10))$$

If this is substituted into ((6,8)), then all the rest of the computation proceeds as before.

We shall now return to our solution on page 44 and improve it by making use of the fourth and fifth observations given on page 24 instead of the first and third. It can be seen from the ephemeris on page 51 that there is some disagreement between the prediction and the observation of September 23. It will be observed that when t_0 is taken as the time of one of the observations there is a considerable simplification of the equations, for $f = 1$, and $g = dg = df = 0$, in fact, the equations reduce to $dy_0 = U_0 dx_0$ and $dz_0 = V_0 dx_0$ (in Case I), or, in general

$$d\rho = \frac{dx}{\cos\delta\cos\alpha} = \frac{dy}{\cos\delta\sin\alpha} = \frac{dz}{\sin\delta} \qquad ((6,11))$$

In this example we shall retain the same t_0 which was used in the preliminary solution. If for any good reason it becomes necessary to change t_0 to some other date than the one for which the preliminary elements are given, this can be accomplished by means of equations ((4,24)). However, such a change should be avoided whenever possible, because it is then necessary to make an adjustment of the components of $\mathbf{r}_0$ in order that the observation at this new epoch, t_0, is satisfied; otherwise the simplification shown in ((6,11)) will not be valid. Furthermore, this means that the preliminary elements to be corrected are now a combination of this artificially adjusted position vector and the velocity vector which was associated with the former orbit at the epoch which has been discarded. This combination usually makes the residuals of the other observations larger than they would be if the epoch were not changed. It is usually better to keep the epoch where the residuals are already small; larger residuals which are associated with relatively larger values

of τ will be absorbed by moderate corrections to x_0'. Also it will be noted that when we have only six equations, it is not necessary to divide through by ρ in the lefthand Cracovian in ((6,7)) or the corresponding one in ((6,4)). In simple situations where the residuals are small and the time intervals are not too long, Stumpff recommends that df and dg be set equal to zero. Then the equations reduce to (Case I)

$$\begin{aligned} f_1(U_0 - U_1)\,dx_0 - U_1 g_1\,dx_0' + g_1\,dy_0' &= (x+X)_1 \Delta U_1 (O-C) \\ f_1(V_0 - V_1)\,dx_0 - V_1 g_1\,dx_0' + g_1\,dz_0' &= (x+X)_1 \Delta V_1 (O-C) \end{aligned} \qquad ((6,12))$$

For the purposes of illustration, we shall make all the necessary computations for the two equations of condition that are derived from the fourth observation by means of the series expressions, and all those corresponding to the fifth observation by means of the closed expressions. The latter depend, in part, upon the elements computed on page 51.

n	τ^n							
0	+1.0	M	+26°.36942	x + X	+1.3221344	+1.5140612	r	2.4024356
1	+0.3606379	E	+32.30902	y + Y	-0.4087065	-0.5054787	F	+0.0626129
2	+0.1300597	cos E	+0.8451777	z + Z	-0.2140977	-0.3521307	G	+0.3519395
3	+0.0469045	cos E - e	+0.6512238	ρ	1.4003279	1.6345905	3τ	+2.5257590
4	+0.0169155	sin E	+0.5344854	tan α	-0.3091263	-0.3338562	L	+0.3595898
5	+0.0061004			sin δ	-0.1528911	-0.2154244		
f	+0.9950379		+0.9733659	α	22 51 17.34	22 46 09.14		
g	+0.3600436		+0.8345093	δ	-8 47 40.2	-12 26 25.5		
				(O - C)	-14.85, -213.5	-152.11, -1809.7		

On pages 78 and 79 are given the numerically evaluated Cracovians ((6,8)), the two pairs of equations, and their solution. The last coefficient in the elimination has a value of about +0.01, so that the solution is moderately determinate. This is, of course, still a short arc compared with a complete revolution. The following computations on the left half of the page show the application of the first solution to the preliminary elements and the residuals which result from the corrected elements. These are not zero because of the neglected second order terms in ((6,1)). The correction, +0.366 which x_0 requires is quite sizeable. These residuals are then inserted directly into the right hand members of the same equations and the right half of the page shows the second result. These residuals are also not zero, but this time it is due mainly to the fact that the coefficients in the equations were not recomputed and, strictly speaking, they do not belong to the elements which are being corrected. In many cases, the difference will be inappreciable. The astute computer will realize that by juggling slightly the values which he inserts into the righthand side of the equation the second time, it is usually possible to obtain corrections which will remove the residuals; e.g. the last declination is over-corrected by 0.07 of its own value, therefore if we use +62″.7 we may expect to reduce the residual to an acceptable amount. In the present case, it is just about as much work to do this as to make a third solution.

If the reader has reproduced the illustrations by his own computations, he is now prepared to appreciate the evaluation of the relative advantages of the two principal methods of determining a preliminary general solution as described in Chapters 4 and 5. In actual practice, the observations of newly discovered minor planets usually extend over two or three months before the object is cut off from view by the Sun. The computer is usually confronted with two separate problems: first, the determination of a preliminary orbit from the first few observations and an ephemeris to facilitate further observations; and second, the determination of a reliable orbit from all the observations in order to insure recovery and further observations in subsequent oppositions.

There is no sharp line of demarcation between the relative advantages of the two methods, and each has its proponents. However, for the cases of ordinary minor planets which do not describe an unusually large heliocentric angle during the interval covered by the observations, the computers who have gained equal facility and experience with both methods will usually prefer the Gaussian method. This is due mainly to the fact that there are only two unknowns to be dealt with and there are no residuals in the observations. The successive solutions needed to satisfy the dynamical conditions converge relatively rapidly. If one wishes to change from one set of basic

		dx_0	dy_0	dz_0
+0.0026834	+0.0003205	+0.9950379	0.0	0.0
+0.0003266	+0.0000589	0.0	+0.9950379	0.0
0.0	0.0	0.0	0.0	+0.9950379
		+0.0061101	-0.0015225	-0.0006301
		+0.0007347	-0.0001687	-0.0000736
+0.0142732	+0.0039645	+0.9733659	0.0	0.0
+0.0039017	+0.0016318	0.0	+0.9733659	0.0
+0.0016233	+0.0008158	0.0	0.0	+0.9733659
		+0.0330406	-0.0066735	-0.0031685
		+0.0093142	-0.0014929	-0.0008337
+0.0000171		+0.2957041	+0.9501437	-0.0000816
-0.4375435		+0.1470544	-0.0455831	+0.9831006
+0.0004106		+0.3290561	+0.9179327	-0.0009768
*		+0.2162176	-0.0733495	+0.9476629
-0.4363224				
-0.0133304				
*				
-0.478679				
*				
	First solution	+0.3658373	-0.0917164	-0.0297507
	Second solution	-0.0151778	+0.0038051	+0.0012343

	+2.6124373		+0.1803109	+2.5972595	+0.1846062
r_0	-0.7391871	r_0'	+0.6142224	-0.7353820	+0.6155815
	-0.2748747		+0.0081133	-0.2736404	+0.0110068
r^2	7.4467823	G^2	0.4098470	7.3614227	0.4131412
r	2.7288793			2.7131942	
a	3.0954520	P	5.4461064	3.0864451	5.4223535
$\sqrt{a}$	1.7593897	n°	0.1809753	1.7568281	0.1817680
e sin E	+0.0084095	e	0.0140947	+0.0135305	0.0148077
e cos E	+0.1184230			+0.1209323	
1 - e cos E	0.8815770	e	0.1187212	0.8790677	0.1216869
		e°	6.80222		6.97215
tan E	+0.0710124	cos E	+0.9974881	+0.1118849	+0.9937988
E°	+4.06189	cos E - e	+0.8787669	+6.38398	+0.8721119
M°	+3.58006	sin E	+0.0708340	+5.60874	+0.1111911
M_1°	+7.37415		+12.43748	+9.41945	+14.50496
E_1°	+8.36356		+14.09390	+10.71584	+16.48320
cos E	+0.9893650		+0.9698979	+0.9825614	+0.9589030
cos E - e	+0.8706438		+0.8511767	+0.8608745	+0.8372161
sin E	+0.1454538		+0.2435117	+0.1859383	+0.2837341
f	+0.9968045		+0.9826567	+0.9967500	+0.9823738
g	+0.3602534		+0.8370546	+0.3602471	+0.8369777

dx'_0	dy'_0	dz'_0		
+0.3600436	0.0	0.0	+0.2963686	+0.1469990
0.0	+0.3600436	0.0	+0.9550736	-0.0456152
0.0	0.0	+0.3600436	0.0	+0.9880843
+0.0007337	-0.0002115	-0.0000801	+0.0474395	+0.1175793
+0.0001323	-0.0000381	-0.0000144	+0.7026314	+0.0903145
+0.8345093	0.0	0.0	+0.3271471	+0.2116580
0.0	+0.8345093	0.0	+0.9449734	-0.0732754
0.0	0.0	+0.8345093	0.0	+0.9745930
+0.0091710	-0.0014578	-0.0008190	+0.1231261	+0.2840584
+0.0038698	-0.0005196	-0.0003310	+0.7036708	+0.0871163

+0.1068334 dx'_0	+0.3438313 dy'_0	-0.0000139 dz'_0	+0.0575072 dx_0	=	-0.0014942	+ 533
+0.0530243	-0.0164518	+0.3557427	+0.0785342		-0.0014494	+ 428
+0.2768596	+0.7880440	-0.0003338	+0.0990074		-0.0176220	+7565
+0.1795728	-0.0616084	+0.8130454	+0.1575404		-0.0143414	+6490
+0.1068365	+0.3438302		+0.0575099		-0.0014944	+ 533
-0.0255466	+0.0105046		+0.0096034		+0.0048256	-2412
+0.2769333	+0.7880187		+0.0990721		-0.0176279	+7568
-0.0139957			+0.0142825		+0.0061970	-2769
-0.0292382			+0.0082827		+0.0050606	-2513
			+0.0103177		+0.0037746	-1566
-0.0694461	-0.0439586	-0.0765187				
+0.0042953	+0.0013591	+0.0028935				

x + X	+1.6658057	+1.8369319	+1.6520811	+1.8248638
y + Y	-0.5169718	-0.6367405	-0.5126531	-0.6317030
z + Z	-0.2717347	-0.4474777	-0.2694471	-0.4437662
ρ	1.7652219	1.9949921	1.7506533	1.9814401
$\tan\alpha$	-0.3103434	-0.3466326	-0.3103074	-0.3461645
$\sin\delta$	-0.1539380	-0.2243005	-0.1539123	-0.2239615
α	22 51 02.07	22 43 31.68	22 51 02.52	22 43 37.43
δ	-8 51 18.7	-12 57 42.3	-8 51 13.4	-12 56 30.5
(O - C)	+0.42, +5.0	+5.35, +67.1	-0.03, -0.3	-0.40, -4.7

observations to another, this can be done with greater facility: it is necessary only to adopt values of the ρ's from the previous solution in order to get started. In the method of La Place it is unlikely that the observations will be well represented at the stage where one has performed about the same amount of computation. For moderate arcs of a month or more, when a differential correction would be required in the La Placian method, the Gaussian method is definitely superior.

As the amount of true anomaly described during the interval covered by the observations approaches one radian, these advantages of the Gaussian method begin to disappear, and the differential correction process which we have just described becomes preferable. This process is at a distinct disadvantage for short arcs because the unknowns contain essentially a factor τ in the denominator. This was strikingly illustrated in the case of the orbit computed from the first three observations of 1936 CA = Adonis. By the same token, the advantage increases as τ increases, and, as we shall see in the next chapter, it requires no modification when the perturbations are taken into account. There is also another group of reasons for preferring this procedure. Whichever set of elements is finally adopted to represent the elliptic orbit, it purports to represent the object's position equally well at any point in the orbit, even though the observations from which the elements are determined are all clustered within only a short arc in one part of the ellipse.

Under such conditions, the set of elements consisting of $\mathbf{r}_0$ and $\mathbf{r}'_0$ will have a more determinate solution than any of the other sets usually used. In general, the solution for these elements is to be preferred for all elliptic orbits so long as the arc is not too short, and no difficulty will be encountered when the orbit is nearly circular. But a baffling situation develops when the computer attempts to correct an initial parabolic orbit, using the closed expressions for df and dg. Both M and N contain the semi-major axis as a factor and this is then infinitely large. There appears to be no simple means by which the formulas can be transformed to circumvent this difficulty. This matter has been discussed by the author in the Astronomical Journal, v.48, p.105. The question has not been properly resolved. It should be recognized that the corrections which can be obtained by setting M = N = 0 may produce an improvement in the residuals, but the results are neither rigorous nor necessarily final.

The determinateness of the solution will be further increased if one of the four degrees of variability can be eliminated by conditioning the period of the solution to some preassigned value. We have already shown how this can be accomplished in the Gaussian method by using Lambert's equation, or if the solution is to be a parabola, the process reduces to Olbers' method. In the La Placian method, we have deferred this topic until now because it is, in principle, a differential correction process and does not flow naturally from the equations for the general solution. In determining either a preliminary orbit or an improved orbit, we must solve the equations ((4,13)) and ((4,12)) or ((6,8)) in such a way as to satisfy ((3,13)) when the semi-major axis has the value corresponding to the adopted period. For a parabola, ((3,13)) reduces to $2\mu = \omega$. In either case, we already have enough equations to solve for all the unknowns. If we are obliged to superimpose the further condition ((3,13)), we must relinquish some of the conditions defined by the other equations. However, if the object is actually moving in an orbit of approximately the same period as the adopted value, there should be very little contradiction among all these equations.

In the case of a preliminary orbit, we start by assuming some value of x_0, instead of using the first equation of ((4,13)). This fixes r_0^2, x'_0, y'_0, and z'_0, and then

$$G^2 + 1/a - 2/r_0 = \Delta(x_0) \qquad ((6,13))$$

By assuming reasonable trial values of x_0 we may solve for $\Delta(x_0) = 0$. In this solution we shall have satisfied exactly the dynamical conditions corresponding to the adopted period; and the equation which we have disregarded is only an approximate equation anyway. It is this change which increases the determinateness of the solution. The validity of the conditioned period which we have imposed is indicated by the residuals which result.

In the case of a differential correction, we disregard one of the equations given by ((6,8)). It is usually best to disregard the equation from the first observation corresponding to the coordinate having the lesser angular motion on the sky. Then the remaining three equations enable us to determine $dx'_0 = a_x + b_x dx_0$ and similarly for dy'_0 and dz'_0. We then solve for dx_0 so as to satisfy ((3,13)) in the same way as we did for x_0 above. It is then the representation of the unused coordinate which indicates the validity of the conditioned period. As an illustration, we now use the same observational data for Comet Oterma II as was used in Chapter 5 and repeat the same problem.

The remaining portion of the computation involves only steps which have already been illustrated. As soon as the elements permit the identification of this comet with 1867 I, the computer proceeds to the determination of a conditioned solution with the same adopted value of the semi-major axis as in Chapter 5, a = 11.3, and the same equations for r_0^2, etc. as have just been used for the parabolic solution. Otherwise when he attempted a differential correction based on the longer arc, he would be faced with the difficulty already mentioned above. The new conditioned solution proceeds exactly as before, except for the computation of $\Delta(y_0)$; and the differential correction for the longer arc is the same as has been illustrated above for (1361), except for the final solution for the unknowns. These computations are therefore left as an exercise for the student.

The evaluation of the relative advantages of Olbers' method and the La Placian method of determining parabolic orbits is somewhat different from the case of minor planets. Olbers' method is reduced to the determination of only one unknown, but the solution is dependent upon the proper choice of the somewhat artificial parameter, M. The La Placian method still contains, at best, four unknowns. Due to the greater curvature and larger inclinations of comet orbits, the solution is usually much better determined from a short time interval than it is for a minor planet

1942 Nov.	11.18242	12.24299	13.12670
JD	674.68242	675.74299	676.62670
	-0.6605904	-0.6465855	-0.6347413
R	-0.6765258	-0.6875155	-0.6964900
	-0.2934346	-0.2982001	-0.3020980
U	+0.5186008	+0.5194436	+0.5202436
V	+0.0393636	+0.0458533	+0.0514353
P	-0.3097436	-0.2894600	-0.2723968
Q	-0.2668041	-0.2666752	-0.2662738
S	1.1271629	1.1277961	1.1284056
τ_1	-0.0182441	+0.0334458	+0.0152017
	(W,1)		(W,3)
U	+0.0461958		+0.0526257
V	+0.3557150		+0.3671958
P	+1.1117896		+1.1224534
Q	+0.0070653		+0.0264049

		$\mathbf{r}_0$	$\mathbf{r}_0'$	$\mathbf{r}_0 \times \mathbf{r}_0'$
		+0.9715986	-0.9839073	+0.5659759
		+1.3132101	+0.1317341	-0.7722361
		+0.3268902	+0.4637784	+1.4200697
r^2	ω	2.7753818	+0.4325595	
r	σ	1.6659477	-0.2274872	
μ	σ^2	0.2162798	+0.0517504	

	W_0'	$\frac{1}{2}W_0''$
U	+0.0497032	+0.1922484
V	+0.3619776	+0.3432658
P	+1.1176065	+0.3188382
Q	+0.0176147	+0.5782370

$$D = +0.0525282$$

$$r_0^2 = +1.2719242\,y_0^2 + 0.3251722\,y_0 + 0.1549028$$

$$y_0' = +0.0327232 \quad +0.4577907/r_0^3$$

$$x_0' = +0.0497032\,y_0 + 0.5194436\,y_0' - 1.1176065$$

$$z_0' = +0.3619776\,y_0 + 0.0458533\,y_0' - 0.0176147$$

y	+1.3	+1.31	+1.3132
y^2	1.69	1.7161	1.7244942
r^2	2.7271786	2.7636275	2.7753448
r	1.6514171	1.6624162	1.6659366
μ	0.2220392	0.2176611	0.2162841
y′	+0.1343707	+0.1323664	+0.1317360
x′	-0.9831943	-0.9837384	-0.9839068
z′	+0.4591175	+0.4626454	+0.4637748
$\Delta(y_0)$	+0.0155657	+0.0037653	+0.0000119

n	$f^{(n)}$	$g^{(n)}$	τ_1^n	
0	+1.0	0.0		
1	0.0	+1.0	-0.0182441	+0.0152017
2	-0.1634451	0.0	+0.0003328	+0.0002311
3	-0.0371817	-0.0544817	-0.0000061	+0.0000035
4	+0.0012104	-0.0185908	+0.0000001	+0.0000001
5	+0.0046744	-0.0002991		
		f	+0.9999458	+0.9999621
		g	-0.0182438	+0.0152015
		x + X	+0.3289057	+0.3218636
		y + Y	+0.6342098	+0.6186729
		z + Z	+0.0249768	+0.0318299
		ρ	0.7148600	0.6981157
		cot α	+0.5186071	+0.5202484
		sin δ	+0.0349394	+0.0455940
		α	4 10 21.22	4 10 03.45
		δ	+2 00 08.2	+2 36 47.7
		(O - C)	(+0.07, -3.4)	(+0.05, -2.4)

For the solution with the conditioned period, we start with the previous computations for $y_0 = 1.3$, which yields $\Delta(y_0) = -0.0729299$. Then

y	+1.25	+1.24	+1.2401980
y^2	+1.5625	+1.5376	+1.5380911
r^2	2.5487496	2.5138270	2.5145160
r	1.5964804	1.5855053	1.5857225
μ	0.2457589	0.2508978	0.2507947
y′	+0.1452293	+0.1475819	+0.1475347
x′	-0.9800391	-0.9793141	-0.9793287
z′	+0.4415165	+0.4380046	+0.4380741
$\Delta(y_0)$	-0.0122449	+0.0002473	-0.0000010

$\mathbf{r}_0$	$\mathbf{r}_0'$
+0.9336725	-0.9793287
+1.2401972	+0.1475349
+0.3235423	+0.4380739

orbit. But the larger eccentricity reduces the interval within which the f and g series will converge, and therefore the closed forms must be used sooner. This involves a greater amount of computation and leads to difficulties already mentioned. Also the neglect of the third and higher derivatives of the observed quantities increases the size of the residuals, as can be seen by comparing the above example with the corresponding result in Chapter 5. This effect may be diminished by using more than three observations, as suggested on page 42 following ((4,10)). An example of this practice will also be found in the Astronomical Journal, v. 45, p. 127.

Olbers' method is essentially in closed form at all times, and its only conspicuous point of disadvantage is that M and all the other coefficients and auxiliaries must be recomputed every time the basic observations are changed. Also it does not admit of a straightforward least squares solution when numerous observations become available. On the whole, the La Placian method admits of slightly better facility and flexibility in dealing with the unpredictable problems that may be presented by a newly discovered comet. But Olbers' method is highly effective in preliminary parabolic orbit determinations.

Another method of obtaining differential corrections, when the observations extend over a sufficiently long arc of the orbit to make the solution fairly determinate, is given by Eckert and Brouwer in the Astronomical Journal, v.46, p.125. The unknown differentials which are determined in this method are dM_0, da/a, de, ψ_x, ψ_y, ψ_z. The last three unknowns are the radian measures of the rotations of the orbit about the x-, y-, and z-axes. These are equivalent to the rotation of the orbit about a vector Ψ whose components are ψ_x, ψ_y, ψ_z, and they take the place of differential corrections to i, Ω, and ω.

An increment, dM_0, is the same as a constant increment to each value of M, i.e. $dM_0 = dM$. Therefore, since $\mathbf{r} = \mathbf{A}(\cos E - e) + \mathbf{B}\sin E$,

$$\frac{d\mathbf{r}}{dM_0} = (\mathbf{B}\cos E - \mathbf{A}\sin E)\frac{dE}{dM} = \frac{\mathbf{B}\cos E - \mathbf{A}\sin E}{1 - e\cos E} = \frac{\mathbf{r}'}{n} \tag{(6,14)}$$

An increment, da, has two effects on $\mathbf{r}$: one is to increase the size of the orbit uniformly in all directions, and the other is to change the value of n, which has the effect of producing a dM which increases linearly with the time. Since

$$\frac{dM}{da} = (t - t_0)\frac{dn}{da} = -\frac{3}{2}\frac{n}{a}(t - t_0),$$

we have

$$\frac{d\mathbf{r}}{da/a} = \mathbf{r} + \frac{d\mathbf{r}}{dM}\frac{dM}{da/a} = \mathbf{r} - 1.5\,n\,(t - t_0)\frac{\mathbf{r}'}{n} = \mathbf{r} + m\frac{\mathbf{r}'}{n} \tag{(6,15)}$$

where $m = -1.5\,k\,(t - t_0)\,a^{-3/2} = 0.02617994\,(M_0 - M)^\circ$.

An increment, de, has more complicated effects: one is direct, another is indirect through the absolute magnitude of $\mathbf{B}$, and another is indirect through the solution of Kepler's equation. Thus $\frac{d\mathbf{r}}{de} = -\mathbf{A} - \frac{e}{1 - e^2}\mathbf{B}\sin E + \frac{d\mathbf{r}}{dE}\frac{dE}{de}$. From $M = E - e\sin E$, we get

$$0 = (1 - e\cos E)\frac{dE}{de} - \sin E \quad \text{and} \quad \frac{d\mathbf{r}}{dE} = \mathbf{B}\cos E - \mathbf{A}\sin E \quad \text{or} \quad \frac{d\mathbf{r}}{dE}\frac{dE}{de} = \frac{\mathbf{r}'}{n}\sin E$$

Also let us eliminate $\mathbf{A}$ and $\mathbf{B}$ in terms of $\mathbf{r}$ and $\mathbf{r}'$ by means of equations ((4,27)). Then

$$\frac{d\mathbf{r}}{de} = -\frac{(\cos E + e)}{1 - e^2}\mathbf{r} + \left[\frac{2}{1 - e^2} - \frac{e(\cos E + e)}{1 - e^2}\right]\sin E\frac{\mathbf{r}'}{n} = H\mathbf{r} + K\frac{\mathbf{r}'}{n} \tag{(6,16)}$$

where $H = -\frac{(\cos E + e)}{1 - e^2}$, $K = \left[\frac{2}{1 - e^2} + eH\right]\sin E$.

The effects of small rotations about the coordinate axes are given by the usual antisymmetric rotation determinant whose non-zero elements are the coordinates themselves taken in cyclical order. Thus, the final equations of condition are obtained from the following Cracovian product:

$$\begin{array}{c} \begin{array}{cccccc} \psi_x & \psi_y & \psi_z & dM_0 & da/a & de \end{array} \\ \left\{\begin{array}{cccccc} 0 & +z & -y & x'/n & x+m\,x'/n & Hx+Kx'/n \\ -z & 0 & +x & y'/n & y+m\,y'/n & Hy+Ky'/n \\ +y & -x & 0 & z'/n & z+m\,z'/n & Hz+Kz'/n \end{array}\right\} \end{array} \left\{\begin{array}{cc} -\sin\alpha/\rho & -\sin\delta\cos\alpha/\rho \\ +\cos\alpha/\rho & -\sin\delta\sin\alpha/\rho \\ 0 & +\cos\delta/\rho \end{array}\right\} = \left\{\begin{array}{c}\cos\delta\,\Delta\alpha \\ \Delta\delta\end{array}\right\} \qquad ((6,17))$$

It is obvious that in this method it is very simple to compute the numerical coefficients of the unknowns, and this method is to be highly recommended for elliptic orbits when the observations are well distributed around the orbit. It has one serious limitation which can arise in the case of a nearly circular orbit which lies nearly in the fundamental plane of the coordinate system. Any rotation of the orbit about the z-axis may be almost equally as well accomplished by a corresponding increase in the mean anomaly, so that the equations are unable uniquely to determine which unknown is to take up the required correction, in other words, these variables are not well separated and the solution is not well determined. This situation is discussed by the authors in the original reference cited above. The modifications required in order to apply the method to nearly parabolic orbits and other topics related to differential corrections are discussed in the author's paper cited above, Astronomical Journal, v. 48, p. 105. We shall not repeat the entire presentation here, but for the sake of convenience and ready reference, we copy the formulas for the correction of an initial parabolic orbit. If the de column of the principal Cracovian is omitted, it will correct the orbit to another parabola. If it is included, it will be necessary to have observations extending over a considerable arc of true anomaly or else the solution will be indeterminate.

$$\begin{array}{c} \begin{array}{cccccc} \psi_x & \psi_y & \psi_z & k\,dT & dq/q & de \end{array} \\ \left\{\begin{array}{cccccc} 0 & +z & -y & -x' & x-1.5\,k\,(t-T)\,x' & dx/de \\ -z & 0 & +x & -y' & y-1.5\,k\,(t-T)\,y' & dy/de \\ +y & -x & 0 & -z' & z-1.5\,k\,(t-T)\,z' & dz/de \end{array}\right\} \end{array} \left\{\begin{array}{cc} -\sin\alpha/\rho & -\sin\delta\cos\alpha/\rho \\ +\cos\alpha/\rho & -\sin\delta\sin\alpha/\rho \\ 0 & +\cos\delta/\rho \end{array}\right\} = \left\{\begin{array}{c}\cos\delta\,\Delta\alpha \\ \Delta\delta\end{array}\right\} \qquad ((6,18))$$

where

$$\mathbf{r} = q\,\mathbf{P}(1-\tan^2\tfrac{1}{2}v) + 2q\,\mathbf{Q}\tan\tfrac{1}{2}v, \qquad \mathbf{r}' = \frac{2q\,\mathbf{Q} - 2q\,\mathbf{P}\tan\frac{1}{2}v}{\sqrt{2}\,q^{3/2}\,(1+\tan^2\frac{1}{2}v)}$$

$$\frac{d\mathbf{r}}{de} = \tfrac{1}{2}\tan^2\tfrac{1}{2}v\,\mathbf{r} + [0.45\,k\,(t-T) - 0.2\sqrt{2}\,q^{3/2}\tan\tfrac{1}{2}v\,(1+\tan^2\tfrac{1}{2}v)^2]\,\mathbf{r}'$$

To illustrate the application of ((6,17)), we return to our solution on page 65 and make a differential correction based on the six observations shown in the following computations. The times of observation have been corrected for "light time" in accordance with the computed value of the geocentric distance. We anticipate the contents of the next chapter by including the values of the perturbations which are to be added to the heliocentric, rectangular coordinates.

Before we can solve the twelve equations of condition for the values of the six unknowns, it is necessary to consider the principle of "Least Squares". No attempt will be made to give a comprehensive treatment of this subject, as it is adequately developed in numerous other places. If we have a large number of linear algebraic equations of the form

$$a_i x + b_i y + c_i z + \ldots\ldots = n_i \qquad ((6,19))$$

and if the values which we adopt for the unknowns are not exact solutions of all the equations, then each equation will have a residual, v_i, of the form

$$v_i = n_i - a_i x - b_i y - c_i z - \ldots \qquad ((6,20))$$

If all the equations are exactly consistent, then we may find a set of values for x, y, z, etc. which will make all the v's vanish. But if this is not the case, then we adopt the principle that we wish to have such a set of values for the unknowns as will cause the sum of the squares of the residuals to be a minimum. If such a set is found, it will satisfy the condition that $\frac{\partial}{\partial x}\sum v_i^2 = 0$, and similarly for y, z, etc. If we square the expression for v and perform this differentiation, we obtain the same conditions expressed in the form

$$\begin{array}{l} (a\ a)\,x + (a\ b)\,y + (a\ c)\,z + \ldots\ldots = (a\ n) \\ (a\ b)\,x + (b\ b)\,y + (b\ c)\,z + \ldots\ldots = (b\ n) \\ (a\ c)\,x + (b\ c)\,y + (c\ c)\,z + \ldots\ldots = (c\ n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots \end{array} \qquad ((6,21))$$

The notation, (a b), means the sum of all the products of the corresponding pairs of factors, a and b. The equations ((6,21)) are called the "Normal Equations", and their solution yields the desired values of the unknowns. These equations are obtained by accumulating each coefficient, one at a time, in the product dials of the computing machine. There are two principal techniques for the solution of these equations. The first is to replace (a n) by the literal term, A, (b n) by B, etc. Then, by elimination, each unknown is determined as a linear combination of A, B, etc. of the form

$$\begin{Bmatrix} x \\ y \\ \cdot \end{Bmatrix} = \begin{Bmatrix} A \\ B \\ \cdot\cdot \end{Bmatrix} \begin{Bmatrix} a_{11} & a_{21} & \dots \\ a_{12} & a_{22} & \dots \\ \cdot & \cdot & \dots \end{Bmatrix} \qquad ((6,22))$$

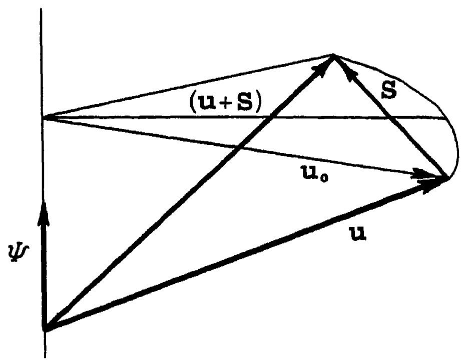
and the unknowns may be found by substitution of the numerical values of A, B, etc. This method has the advantage of producing directly the "weight" of each unknown in the solution, e.g. $1/a_{11}$ is the weight of x, $1/a_{22}$ is the weight of y, etc. This method is described in Newcomb's Compendium of Spherical Astronomy, p. 78.

The second method is designed to obtain the numerical solution with the minimum amount of arithmetic work. This method is described in Doolittle's Practical Astronomy, p. 53, and the solution is here arranged in a pseudo-Cracovian form:

$$\left\{\begin{array}{cccccc} (a\ a) & (a\ b) & (a\ c) & (a\ d) & \dots & (a\ n) \\ 0 & (b\ b) & (b\ c) & (b\ d) & \dots & (b\ n) \\ 0 & 0 & (c\ c) & (c\ d) & \dots & (c\ n) \\ 0 & 0 & 0 & (d\ d) & \dots & (d\ n) \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \\ (a\ a) & (a\ b) & (a\ c) & (a\ d) & \dots & (a\ n) \\ 0 & (bb1) & (bc1) & (bd1) & \dots & (bn1) \\ 0 & 0 & (cc2) & (cd2) & \dots & (cn2) \\ 0 & 0 & 0 & (dd3) & \dots & (dn3) \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \dots & \cdot\cdot \end{array}\right\} \left\{\begin{array}{cccccc} +1 & -(a\ b)/(a\ a) & -(a\ c)/(a\ a) & -(a\ d)/(a\ a) & \dots \\ 0 & +1 & 0 & 0 & \dots \\ 0 & 0 & +1 & 0 & \dots \\ 0 & 0 & 0 & +1 & \dots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & -(bc1)/(bb1) & -(bd1)/(bb1) & \dots \\ 0 & 0 & 0 & -(cd2)/(cc2) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \dots \end{array}\right\} \qquad ((6,23))$$

The upper half of the lefthand Cracovian consists of the coefficients of the normal equations. The lower half of this Cracovian contains the coefficients of the elimination equations. These are given by the product of the two Cracovians and are obtained one line at a time. This is only a pseudo-Cracovian multiplication because of the zeros in the product which are not obtained as a result of the formal process. It is this second method which is used in the illustration.

Once we have determined the values of the six unknowns, it is a simple matter to apply dM_0, de, and $(1 + da/a)$. The vector Ψ is not so simple. This problem is discussed by the author in the Astronomical Journal, v. 53, p. 15. Consider any vector $\mathbf{u}$ which is to be rotated about Ψ. The terminal point of $\mathbf{u}$ moves through an arc of a circle from the initial position to the rotated position. Let $\mathbf{S}$ represent the total change in position of the terminal point of $\mathbf{u}$ due to the rotation, or $\mathbf{S}$ is the vectorial difference between the rotated and unrotated positions of $\mathbf{u}$. Then $(\mathbf{u} + \mathbf{S})$ is the final, rotated position of $\mathbf{u}$. Since the rotation produces no change in absolute magnitude, it is evident that $\mathbf{u}^2 = (\mathbf{u} + \mathbf{S})^2$ or $\mathbf{S}\cdot(\mathbf{u} + \frac{1}{2}\mathbf{S}) = 0$. Since $\mathbf{S}$ lies in a plane perpendicular to Ψ, we have



$$\mathbf{S}\cdot\Psi = 0.$$

If $\mathbf{S}$ is perpendicular to each of two vectors, it must be proportional to their "cross product".

$$\mathbf{S} = k\ \Psi \times (\mathbf{u} + \tfrac{1}{2}\mathbf{S})$$

To find k, let $\mathbf{u}_0$ be the component of $\mathbf{u}$ which is perpendicular to Ψ. Then it is evident from the figure at the right that

$$|\mathbf{S}| = 2\,u_0 \sin\tfrac{1}{2}\psi$$

$$|\mathbf{u}_0 + \tfrac{1}{2}\mathbf{S}| = u_0 \cos\tfrac{1}{2}\psi$$

Therefore, if we take the absolute magnitude of both sides of the equation for **S**, we have

$$2u_0 \sin\tfrac{1}{2}\psi = k\psi u_0 \cos\tfrac{1}{2}\psi \quad \text{or} \quad k = \frac{\tan\frac{1}{2}\psi}{\frac{1}{2}\psi} = 1 + \frac{\psi^2}{12} + \frac{\psi^4}{120} + \dots$$

By repeated substitutions for **S** and expansion of all the parentheses, we obtain

$$\begin{aligned}\mathbf{S} &= k\Psi\times(\mathbf{u}+\tfrac{1}{2}\mathbf{S})\\ &= k\Psi\times(\mathbf{u}+\tfrac{1}{2}[k\Psi\times(\mathbf{u}+\tfrac{1}{2}\mathbf{S})])\\ &= \dots\dots\dots\dots\\ &= \tfrac{1}{2}k\Psi\times 2\mathbf{u} + \tfrac{1}{2}k\Psi\times(k\Psi\times\mathbf{u}) + \tfrac{1}{2}k\Psi\times[\tfrac{1}{2}k\Psi\times(k\Psi\times\mathbf{u})] + \dots\\ &= \quad \mathbf{S}_1 \quad + \quad \mathbf{S}_2 \quad + \quad \mathbf{S}_3 \quad + \dots\end{aligned} \tag{(6,24)}$$

Each succeeding $\mathbf{S}_{i+1}$ is obtained by operating with $\frac{1}{2}k\Psi\times$ upon the preceding $\mathbf{S}_i$. In practice, the numerical calculations may be conveniently arranged as shown in the example.

For the present purpose, we shall let **u** be each of the vectors **A** and **B** in turn. As soon as we obtain the components of **A** after the rotation, they are multiplied by (1 + da/a); those for **B** must be adjusted so that the new absolute magnitude is the same as the semi-minor axis corresponding to the corrected values of a and e.

Observations of 1935 QA = (1361) Leuschneria

	Date	α (1950.0)	δ (1950.0)	X	Y	Z
1	1935 Aug. 30.0006 UT	$23^h\,06^m\,06^s\!.36$	$-3°\,41'\,27''\!.4$	-0.9217386	+0.3782763	+0.1640270
2	1935 Sept. 23.8717	22 51 02.49	- 8 51 13.7	-1.0032412	-0.0014225	-0.0006612
3	1935 Oct. 21.8510	22 43 37.03	-12 56 35.2	-0.8811272	-0.4245110	-0.1841615
4	1936 Dec. 20.94820	5 18 18.15	- 7 20 07.8	-0.0155810	-0.9023277	-0.3913806
5	1938 Feb. 21.98328	9 43 31.55	+ 4 00 27.7	+0.8803933	-0.4139426	-0.1795592
6	1939 April 20.91371	14 10 26.75	+ 9 31 29.7	+0.8696871	+0.4619699	+0.2003330

	1	2	3	4	5	6
	1935 Aug.30	1935 Sept. 23	1935 Oct. 21	1936 Dec. 20	1938 Feb. 21	1939 April 20
JD	8044.4907	8069.3616	8097.33955	8523.435095	8951.46894	9374.40032
M°	+5.22193	+9.73934	+14.82109	+92.21466	+169.96030	-113.22085
E°	+5.94296	+11.07733	+16.83832	+99.09105	+171.04436	-119.29418
Cos E	+0.9946255	+0.9813688	+0.9571260	-0.1580039	-0.9878092	-0.4892938
(Cos E - e)	+0.8730831	+0.8598264	+0.8355836	-0.2795463	-1.1093516	-0.6108362
Sin E	+0.1035383	+0.1921337	+0.2896721	+0.9874385	+0.1556697	-0.8721190
1 - e cos E	+0.8791108	+0.8807221	+0.8836686	+1.0192042	+1.1200607	+1.0594699
ξ	+5	+13	+27	+2425	-6568	-100559
η	+39	+94	+184	+6717	+43162	+71156
ζ	+13	+33	+67	+2138	+7647	+17294
x + X	+1.6647843	+1.6541931	+1.8271930	+0.4036517	-2.0557466	-1.9271821
y + Y	-0.3988686	-0.5133015	-0.6325495	+2.2159557	+1.3851885	-1.2350309
z + Z	-0.1104332	-0.2697905	-0.4443757	-0.2901708	+0.1736290	+0.3838031
ρ	1.7154587	1.7528891	1.9839916	2.2710336	2.4849524	2.3209130
tan α	-0.2395918	-0.3103033	-0.3461865	+0.1821569	-0.6738129	+0.6408481
sin δ	-0.0643753	-0.1539119	-0.2239807	-0.1277704	+0.0698722	+0.1653673
α	23 06 06.33	22 51 02.57	22 43 37.16	5 18 42.33	9 44 06.58	14 10 36.89
δ	-3 41 27.5	-8 51 13.3	-12 56 34.6	-7 20 26.8	+4 00 23.9	+9 31 06.9
(O - C)	(+0.03, +0.1)	(-0.08, -0.4)	(-0.13, -2.6)	(-24.18, +19.0)	(-35.03, +3.8)	(-10.14, +22.8)

	r	ψ_x	ψ_y	ψ_z
1935 Aug. 30	+2.5865224	0.0	-0.2744589	+0.7771410
	-0.7771410	+0.2744589	0.0	+2.5865224
	-0.2744589	-0.7771410	-2.5865224	0.0
	-1.1329038 (= H)			
1935 Sept. 23	+2.6574330	0.0	-0.2691326	+0.5118884
	-0.5118884	+0.2691326	0.0	+2.6574330
	-0.2691326	-0.5118884	-2.6574330	0.0
	-1.1194484			
1935 Oct. 21	+2.7083175	0.0	-0.2602209	+0.2080569
	-0.2080569	+0.2602209	0.0	+2.7083175
	-0.2602209	-0.2080569	-2.7083175	0.0
	-1.0948421			
1936 Dec. 20	+0.4189902	0.0	+0.1009960	-3.1176117
	+3.1176117	-0.1009960	0.0	+0.4189902
	+0.1009960	+3.1176117	-0.4189902	0.0
	+0.0370082			
1938 Feb. 21	-2.9354831	0.0	+0.3524235	-1.7948149
	+1.7948149	-0.3524235	0.0	-2.9354831
	+0.3524235	+1.7948149	+2.9354831	0.0
	+0.8792557			
1939 April 20	-2.7868133	0.0	+0.1817407	+1.7041164
	-1.7041164	-0.1817407	0.0	-2.7868133
	+0.1817407	-1.7041164	+2.7868133	0.0
	+0.3732655			

ψ_x	ψ_y	ψ_z	dM_0	da/a	de
+0.1555883	-0.0372773	+1.5718282	+2.0272554	-0.5133173	+0.4983006
-0.4544824	-1.5146617	+0.0057455	+0.0393802	-0.0667115	+0.0739608
+0.1466387	-0.0455034	+1.5344677	+1.9782795	-0.4773866	+0.5291192
-0.2955495	-1.5205355	-0.0262266	+0.0211065	+0.0775574	-0.0868780
+0.1239430	-0.0429087	+1.3242764	+1.7070983	-0.4386123	+0.5575812
-0.1118145	-1.3581619	-0.0778346	-0.0268905	+0.1811839	-0.1995865
-0.0080465	-0.0437374	+1.3834959	+1.3239195	-3.2251787	+2.6621390
+1.3559487	-0.1819553	-0.0085465	+0.1445320	-0.1386487	+0.2985242
+0.1174109	-0.0795486	+1.3830880	+1.1008892	-4.9135941	+0.4230438
+0.7260583	+1.1866069	+0.0045182	+0.0152868	-0.0243171	+0.0444844
+0.0659604	+0.0422020	+1.4071504	+1.2441946	-8.1571207	-2.2626291
-0.7311054	+1.1951019	-0.0047391	-0.0961921	+0.4728144	+0.1160157

Normal Equations

+3.2859522	+1.0067261	+0.8867784	+1.2740383	-1.8574326	+0.4698741
+1.0067261	+9.3355675	-0.1582480	-0.5101474	+0.5458357	+0.1161121
+0.8867784	-0.1582480	+12.3929036	+13.5888614	-24.8741604	+3.4331947
+1.2740383	-0.5101474	+13.5888614	+15.4833744	-22.6337831	+4.2229178
-1.8574326	+0.5458357	-24.8741604	-22.6337831	+102.0542375	+7.0036047
+0.4698741	+0.1161121	+3.4331947	+4.2229178	+7.0036047	+13.3820159

+0.8620934	-0.3890909	+8.0722228	+10.4636110		+5.7761906
+0.1214444	+0.2934419	+0.6841354			+9.7127741
+1.0450485	+9.3093142	+0.2542888			
+3.0622846		-0.2681095			
		+0.0242294			

-120.14	+65.13	-1557.35	+1202.08	-66.595	-3.49
-0.00058246	+0.00031576	-0.00755025	+0°.33391	-0.00032286	-0.0000169

dM_0	da/a	de		
+1.0507127	+2.3667310	-2.7244205	+0.1358212	+0.0364934
+3.3243545	-1.4725401	+1.5317492	+0.5668909	-0.0087434
+0.0517465	-0.2852834	+0.3210739	0.0	+0.5817254
+0.1959247 (= K)	-0.2091832 (= m)		0.5829345 (= $1/\rho$)	
+0.7469631	+2.4128411	-2.7030487	+0.1690744	+0.0838610
+3.3990347	-1.6248972	+1.8098985	+0.5448568	-0.0260229
+0.0832341	-0.2963875	+0.3315679	0.0	+0.5636890
+0.3638874	-0.3274485		0.5704867	
+0.3999442	+2.5241479	-2.7454169	+0.1648934	+0.1066877
+3.4456294	-1.7947286	+2.1211091	+0.4762991	-0.0369350
+0.1174704	-0.3143146	+0.3494489	0.0	+0.4912273
+0.5494844	-0.4604882		0.5040344	
-2.9192253	+7.6780646	-5.8490135	-0.4330604	+0.0101724
+0.7495572	+1.2537297	+1.6211852	+0.0796719	+0.0552928
+0.3040425	-0.6550495	+0.6145378	0.0	+0.4367231
+2.0089301	-2.4866441		0.4403282	
-1.4693026	+3.7087285	-3.0697944	-0.2257187	+0.0232842
-2.3089701	+12.2360174	+0.8100371	-0.3331529	-0.0157756
+0.0325656	+0.2051613	+0.3207031	0.0	+0.4014334
+0.3326436	-4.5220172		0.4024174	
+1.7550013	-14.2524597	-4.2166985	+0.2322101	+0.0600567
-2.3052645	+13.3564693	+3.5363422	-0.3629369	+0.0384248
-0.2659583	+1.9192801	+0.5492107	0.0	+0.4249252
-1.8099572	-6.5331270		0.4308651	

	n	v
=	+0.″4	0.″0
=	+0.1	+1.6
=	-1.2	+1.1
=	-0.4	+1.8
=	-1.9	-1.2
=	-2.6	-5.1
=	-359.7	-0.2
=	+19.0	-1.5
=	-524.2	-0.1
=	+3.8	+0.9
=	-150.0	-0.1
=	+22.8	-2.7

=	-56.68	+1.0 (5)					
=	+83.51		+1.0 (4)				
=	-1437.50			+1.0 (6)			
=	-1244.06				+1.0 (2)		
=	+4968.12	+0.0182004	-0.0053485	+0.2437347	+0.2217819	+1.0 (1)	-0.0686263
=	-832.38						+1.0 (3)
=	-142.22	-0.0823897	+0.0371851	-0.7714567			-0.5520265
=	-1094.81	-0.0125036	-0.0302120	-0.0704367			
=	+84.73	-0.1122584		-0.0273155			
=	+49.64			+0.0875522			
=	-37.73						

$\frac{1}{2}k\,\Psi\,x$			$2u$			
0.0	-0.00377514	-0.00015788	+5.6352016	-0.00933717	-8103	+13
+0.00377514	0.0	-0.00029123	-2.4469096	-0.02145766	+3520	+31
+0.00015788	+0.00029123	0.0	-0.6317608	-0.00017707	+ 772	0
0.0	-0.00377514	-0.00015788	+2.4439802	+0.02122721	-3512	-30
+0.00377514	0.0	-0.00029123	+5.6218174	-0.00921888	-8072	+13
+0.00015788	+0.00029123	0.0	+0.0257086	-0.00202310	- 67	+ 3

B_0	A	B				
+1.24318189	+2.8072761	+1.2427830	a^2	9.5293401	e	0.1215256
+2.80160923	-1.2444750	+2.8007105	a	3.0869629	e°	6°.96290
+0.01083056	-0.3159477	+0.0108271	P	5.4237182	n	0.18172231
					M	357.56562

JD	8044.4907	8069.3616	8097.33955	8523.435095	8951.46894	9374.40032
M	+5.55971	+10.07931	+15.16353	+92.59459	+170.37789	-112.76604
E	+6.32705	+11.46309	+17.22547	+99.46274	+171.41704	-118.86393
Cos E	+0.9939090	+0.9800529	+0.9551468	-0.1644062	-0.9888008	-0.4827311
(Cos E - e)	+0.8723834	+0.8585273	+0.8336212	-0.2859318	-1.1103264	-0.6042567
Sin E	+0.1102035	+0.1987367	+0.2961327	+0.9863927	+0.1492413	-0.8757686
x + X	+1.6642420	+1.6538699	+1.8271091	+0.4078441	-2.0517817	-1.9250745
y + Y	-0.3987310	-0.5132249	-0.6325314	+2.2167794	+1.3901287	-1.2317065
z + Z	-0.1104060	-0.2697559	-0.4443292	-0.2901475	+0.1736264	+0.3834939
ρ	1.7148986	1.7525563	1.9838982	2.2725919	2.4844340	2.3173434
tan α	-0.2395872	-0.3103176	-0.3461925	+0.1839804	-0.6775227	+0.6398227
sin δ	-0.0643805	-0.1539214	-0.2239678	-0.1276725	+0.0698857	+0.1654886
α	23 06 06.39	22 51 02.39	22 43 37.09	5 18 18.07	9 43 31.55	14 10 26.89
δ	-3 41 28.8	-8 51 15.3	-12 56 31.9	-7 20 06.4	+4 00 26.7	+9 31 32.2
(O - C)	(-0.03,+1.4)	(+0.10,+1.6)	(-0.06,-3.3)	(+0.08,-1.4)	(0.00,+1.0)	(-0.14,-2.5)

Several details of this computation should be noted. Although the preliminary orbit was based on the first three observations, their residuals are now no longer exactly zero, due to the perturbations since the epoch. It is rather unusual that in 1939 the residual in right ascension has become smaller than it was in 1936 and 1938. The residuals remaining in the observations, after the solution, have been computed by substitution of the unknowns into each equation, and these are shown in the v column. They agree fairly well with those obtained directly by recomputation from the corrected elements; the differences are probably due mainly to the effects of neglected second order terms in the equations of condition.

The order in which the unknowns have been eliminated in the solution of the Normal Equations is indicated by the number in parentheses following the unit multiplying factors in the principal diagonal. This order has been chosen to insure that none of the remaining multiplying factors shall become greater than unity. With observations extending over four oppositions, it may be surprising that the value of the final coefficient in the elimination equations, +0.024, is so small. This is due mainly to the fact that the orbit plane nearly coincides with the fundamental plane of the coordinate system in which Ψ is expressed. It will be seen that ψ_z and dM_0 are nearly equal and of opposite sign. In spite of the magnitude of the eccentricity, the type of indeterminacy described on page 83 is present to a certain extent.

CHAPTER 7

SPECIAL PERTURBATIONS

Οὗτοι δὲ οἱ ἀριθμοί, ἄλλος παρ' ἄλλου διαδοχῇ
προσταχθέντες, εἰς ἀπειρίαν φέρουσιν.

In the preceding chapters we have examined various methods of using observed positions of an object in the sky, made from the Earth, in order to deduce numerical quantities which will enable us to describe the motion of the object about the Sun. The equations ⟨⟨3,3⟩⟩ completely define the conditions which the law of gravitation imposes upon such motion, but thus far it has been used only in a simplified form by setting $m_i = 0$. We shall now consider several numerical methods of dealing with the complete equation. For this purpose we return to the results of Chapter 1, in particular, equations ⟨⟨1,10⟩⟩, ⟨⟨1,11⟩⟩, ⟨⟨1,14⟩⟩, ⟨⟨1,15⟩⟩, ⟨⟨1,19⟩⟩, ⟨⟨1,20⟩⟩, and ⟨⟨1,21⟩⟩.

When an object is newly discovered it is not necessary to deal with the complete equations ⟨⟨3,3⟩⟩, unless the object happens to be in very close proximity to one of the bodies m_i, and the Two Body solution which we obtained by the methods of Chapter 4 and 5 is entirely satisfactory for a short time. However, it is easy to see that as time goes on, the object will not continue to follow the elliptical orbit exactly, since that path represents only the effect of the Sun's gravitational attraction, and the other forces due to the major planets, even though they are very small, will cause a slight deviation or "perturbation" which gradually becomes larger and larger. Therefore, we wish to establish methods for computing the exact path of the object, taking into account all the effects of the equations ⟨⟨3,3⟩⟩.

The method which is simplest in principle, most general in application, and most complete in its results is that of computing the numerical value of ⟨⟨3,3⟩⟩ at equal intervals of the time, say 10, 20, or 40 days, place them in tabular form, and solve for the coordinates, x, y, z, in the same manner as we did the example on pages 9 to 11. It is assumed that the positions of the disturbing planets, m_i, are known; and the computations are greatly facilitated by the use of the volumes of Planetary Coordinates. The quantities in the difference columns of the computed functions are automatically determined, but the problem of determining the starting quantities in the first and second summation columns is more complicated. These correspond to what the mathematicians call the "arbitrary" constants of integration. They are, however, certainly not arbitrary, but very greatly restricted, if we wish to have the results of our computation agree with the observed positions of the object on the sky within the limits of observational error. Such complete agreement must be reached by successive approximations. The reader should now refer again to the discussion on pages 9 and 17. To begin, we may choose one of the tabular dates, t_0, and take from our Two Body solution the values of x_0, x_0', y_0, y_0', z_0, z_0'. Then by using these as the values of the left member of ⟨⟨1,10⟩⟩ or ⟨⟨1,11⟩⟩, as the appropriate case may be, we may determine ${}^{II}f_0$ and ${}^{I}f_0$. To obtain the higher order differences which it is necessary to have on the right hand side, use values of x, y, z taken from the Two Body solution for about three or four tabular dates on either side of t_0, so as to be able to evaluate ⟨⟨3,3⟩⟩ for these dates. After the table of functions, differences, and summation columns is set up for these dates, it becomes possible by means of ⟨⟨1,10⟩⟩ to obtain more accurate values of x, y, z from the integration table itself, and these are used to recompute ⟨⟨3,3⟩⟩, i.e., the function column. This will change the differences slightly, and perhaps also the starting values in the summation columns.

The reader who is unfamiliar with this process may take the following view of the problem. A completely blank sheet of paper is laid out to contain certain quantities in certain positions, but at the beginning none of these quantities is known. Furthermore, with only one exception, all of these quantities must be obtained, at least in theory, by successive approximations. Therefore at

every stage we must be willing to enter and use tentative values for every quantity which we are to enter onto the sheet. Only f_0, which is a function of the known x_0, y_0, z_0, can be computed directly. The crux of the computation, once the starting values have been determined, is to enter onto the computing sheet computed values of f_i in the function column which, taken in combination with the differences and summation columns which these f's determine, will yield values of x, y, z by means of ((1,10)) that in turn produce the same computed values of f_i. This is the test that the tentative values at any stage are actually the final values. It is an interesting game to creep up upon the final values this way, especially for one who is skilled in the use of a computing machine and for whom the computations are not a laborious distraction.

This process depends for its convergence upon the size of the interval that is adopted. As explained on page 9, the double integral will be obtained directly from the integration table if all the quantities in the function column are multiplied by the square of the interval. In the problem of motion in the solar system, the interval is $h = wk$, where w is the number of mean solar days in the interval. If the four inner major planets, which have relatively small masses, are not separately included in the perturbation computations, their average secular effect will be taken into account approximately by adding their mass to the mass of the Sun, and using $k = 0.01720215$. It is desirable to keep the interval large in order to reduce the accumulation of numerical errors, but it is more important to keep it sufficiently small so that the convergence of the differences will be reasonably rapid and so that the highest order differences will afford a check by inspection.

We shall now illustrate this process by computing a detailed numerical example. We shall suppose that we have just completed our solution on page 65 and we wish to project the perturbed path of this minor planet into the future. In ordinary cases it is not necessary to do this simply for the sake of having a sufficiently accurate finding ephemeris for the next opposition; the uncertainty of the mean motion will usually be much greater than the neglected perturbations. But if it is intended to obtain accurate positions at four or five oppositions so that reliable elements may be derived, it is then necessary to make an accurate computation of the perturbations before comparison is made with the observations to determine the residuals. These perturbations should not be computed if the residuals are too large, for then the differential correction will produce such large changes in the orbit as to render the perturbations invalid.

It should be observed that in the differential correction process, no account is taken of the differential effect on the observations that is produced by the changes in the perturbations that would accompany the corrections to the elliptic elements. In practically all cases this is a relatively small effect, but whenever it becomes appreciable within the limits of accuracy that is to be attained, it simply imposes one further step in the process of successive approximations by means of which we arrive at the final result. After the corrections are obtained, the perturbations must be recomputed to permit the determination of new residuals, and then the correction is repeated.

In extreme cases of highly disturbed objects such as Jupiter's outer satellites or a comet which passes very close to a perturbing planet, these repetitions may not converge with sufficient rapidity to be practical. In that case, it may be advisable to resort to a numerical process for determining the partial differential coefficients with respect to each of the variables. To do this for any one of the six variables, say w_i, compute the residuals with respect to the given elements to be corrected, and then compute them again with the same elements except that w_i is replaced by $w_i + \epsilon$, where ϵ may be +0.001 or +0.01, say. Then

$$\frac{\partial F}{\partial w_i} = \frac{\Delta F(w_i + \epsilon) - \Delta F(w_i)}{(w_i + \epsilon) - w_i} \qquad ((7,1))$$

This computation must then be repeated for each of the other variables to be corrected. It should be noted that, even though this is a great deal of work and should be used only as a last resort, it is a process which automatically takes into account terms of all orders in the correction.

For nearly all minor planets, only the perturbations due to Jupiter and Saturn need to be taken into account for most purposes. Uranus and Neptune are at such great distances that their direct and indirect terms practically cancel each other, and the inner planets have such small masses and short periods that their effects are also not very large. Only the most refined problems justify the inclusion of these planets. The outer planets need to be included in long-period comet orbits and it is well to include the inner planets in the region around perihelion.

Let us adopt 1935 July 17.0 UT = JD 2428000.5 as the epoch of osculation and an interval of 20 days for the integrations. Solve Kepler's equation for the seven dates from 2427940.5 to 8060.5, compute x, y, z in the elliptic orbit for each date, and also the velocity components at the epoch by means of

$$(wk)\,\frac{\mathbf{B}\cos E - \mathbf{A}\sin E}{P\,(1 - e\cos E)}. \tag{(7,2)}$$

The arrangement of the computations for each date and the numerical results are shown below. The comma (,) represents the units of the 7th decimal place, corresponding to the decimal point which is used in Planetary Coordinates. The quantities P represent the sum of the four planetary terms. If there appears to be an error in the computations, the P's may be differenced separately to determine whether they contain the error or whether it is in the much larger solar term where it is more easily masked. The indirect terms, X(Jup), X(Sat), etc. are copied from Planetary Coordinates. The numerical value to be placed in the f column of the integration table is accumulated in the product dials of the computing machine in one continuous operation. One starts at the bottom of the block, adds in the indirect terms, multiplies the direct terms, copies down P, and then computes the solar term. Each multiplying factor which is constant for the x, y, z columns is in the right hand column and on the line below the factors which it multiplies. In hand work, these may be written in red pencil. They are readily derived as functions of ρ^2 from the tables in Astronomische Nachrichten, v. 260, pp. 325 - 376.

Using the preliminary values of x, y, z which we have obtained from the elliptic orbit, we fill in all the f's and their differences. Those shown in the table for these dates are not these original values; these have since been replaced by the ones which were subsequently improved. Then using the given values of x_0, x_0', etc. and the differences which exist at this stage, substitute into ((1,10)) and ((1,11)) to find ${}^{II}f_0$ and ${}^{I}f_{1/2} = {}^{I}f_0 + \frac{1}{2}f_0$. Next, fill in the first and second summation columns and use ((1,10)) to recompute x, y, z at each of the dates adjoining t_0, then each of the next adjoining dates, etc. It will be seen that the differences are changed only very slightly, so that a recomputation of the starting values gives them finally.

The table is then extended one step at a time, using the extrapolation formulas ((1,12)) to get preliminary values of the coordinates. These are always sufficiently accurate for the planetary terms, and the check formula shows whether or not they are final for the solar terms. If not, the recomputation is readily performed as the P's are directly available. These coefficients of ((1,12)) should be placed on a separate slip of paper or "stencil" as shown in the drawing under the bottom of the integration table; such kindergarten practices make for greater facility of the routine work and reduce the likelihood of numerical errors.

Beginning at JD 2428240.5, the alternate attractions are computed for a 40 day interval in order that we may double the interval. After such values are set down in the f columns of the new tables at 8240 to 8400, we may form the differences and compute ${}^{II}F$ at 8280, 8320, and 8360 by means of ((1,23)). These are checked by having their second differences agree with the computed functions at 8320. Due to rounding-off errors, this cannot be made to agree exactly every time, as can be seen by examining carefully the computations for the y-table. The reader may also notice the raggedness of the differences in the x-table with 20 day interval near 8320. This is due to the rounding-off errors in f at 8320 and 8340 being nearly a full half unit in the last place and of opposite sign.

These integrations have been carried far enough so that the student may represent the 1935 and 1936 observations on page 85 by means of ((1,20)). As a further exercise, the tables may be extended to 1938 and 1939. The work proceeds very smoothly, due to the moderate eccentricity and large distance from the Sun. For more eccentric orbits or smaller heliocentric distances it would be necessary to use a smaller interval and progress would not be so rapid.

About a century ago, in the era of lead pencil and logarithmic computing, Encke proposed a method of computing special perturbations which, so far as the numerical integrations are concerned, is no different in principle from the example which we have just completed, but which was intended to deal with fewer significant figures. For this exposition we shall adopt the notation that x, y, z are the heliocentric coordinates of the object in space at any time t, and x_0, y_0, z_0 are the coordinates which the object would have at the time t if it remained in its Two Body elliptic orbit.

JD	M	E	2427940.5	346.33365	-15.53099
Cos E	Cos E - e	Sin E	+0.9634857	+0.8419433	-0.2677595
- x	- y	- z	-2.0450614	+1.7827200	+0.2693932
r^2		h	7.4329394		0.0058409,646
$(x_j - x)$	$(y_j - y)$	$(z_j - z)$	-5.57441	-2.01229	-1.27240
ρ_j^2		$m_j(wk)^2/\rho_j^3$	36.74236		5,074
$(x_s - x)$	$(y_s - y)$	$(z_s - z)$	+6.69828	-2.07930	-1.70489
ρ_s^2		$m_s(wk)^2/\rho_s^3$	52.09709		0,899
X (Jup)	Y (Jup)	Z (Jup)	+25,23	+27,13	+11,02
X (Sat)	Y (Sat)	Z (Sat)	-3,18	+1,40	+0,72
P_x	P_y	P_z	-0,21	+16,45	+3,75
2427960.5	349.96634	-11.41149	2427980.5	353.59902	-7.28390
+0.9802315	+0.8586891	-0.1978538	+0.9919301	+0.8703877	-0.1267859
-2.1776679	+1.6067134	+0.2737854	-2.2974742	+1.4212625	+0.2765679
7.3987239		0.0058815,290	7.3748646		0.0059100,940
-5.59318	-2.27000	-1.30585	-5.59657	-2.53426	-1.33972
38.14181		4,798	39.53892		4,546
+6.60861	-2.16217	-1.66385	+6.53063	-2.25400	-1.62416
51.11710		0,925	50.36754		0,946
+24,48	+27,78	+11,32	+23,70	+28,41	+11,61
-3,20	+1,37	+0,70	-3,22	+1,34	+0,69
+0,56	+16,26	+4,22	+1,22	+16,10	+4,67
2428000.5	357.23171	-3.151088	2428020.5	0.86440	0.98399
+0.99848805	+0.87694565	-0.05496912	+0.9998525	+0.8783101	+0.0171730
+0.09922103	+0.19767424	-0.00032680 = $(wk)\,r'_0$			
-2.40371104	+1.22741654	+0.27771653	-2.4957125	+1.0263003	+0.2772199
7.36150460		0.0059261,902	7.3587246		0.0059295,488
-5.58392	-2.80391	-1.37401	-5.55463	-3.07780	-1.40869
40.92998		4,316	42.31117		4,106
+6.46509	-2.35375	-1.58587	+6.41266	-2.46031	-1.54898
49.85251		0,961	49.57467		0,969
+22,90	+29,03	+11,90	+22,09	+29,63	+12,17
-3,24	+1,31	+0,68	-3,26	+1,28	+0,67
+1,77	+15,98	+5,13	+2,24	+15,89	+5,55
2428040.5	4.49708	5.11835	2428060.5	8.12977	9.24904
+0.9960125	+0.8744701	+0.0892133	+0.9869991	+0.8654567	+0.1607261
-2.5729260	+0.8191009	+0.2750801	-2.6349180	+0.6070535	+0.2713123
7.3665435		0.0059201,108	7.3849172		0.0058980307
-5.50825	-3.35466	-1.44374	-5.44441	-3.63319	-1.47912
43.67895		3,915	45.02947		3,740
+6.37387	-2.57250	-1.51350	+6.34916	-2.68909	-1.47941
49.53466		0,970	49.73169		0,966
+21,25	+30,22	+12,44	+20,40	+30,79	+12,71
-3,28	+1,24	+0,66	-3,30	+1,21	+0,64
+2,59	+15,83	+5,98	+2,87	+15,81	+6,39

2428080.5 = 1935 Oct. 5.0 UT

-2.6813800	+0.3914255	+0.2659447
7.4137392		0.0058636,699
-5.36290	-3.91208	-1.51476
46.35956		3,580
+6.33882	-2.80885	-1.44670
50.16322		0,952
+19,52	+31,34	+12,96
-3,32	+1,18	+0,63
+3,04	+15,84	+6,79

2428100.5 = 1935 Oct. 25.0 UT

-2.7121302	+0.1735008	+0.2590179
7.4528430		0.0058175,818
-5.26363	-4.19000	-1.55060
47.66626		3,434
+6.34302	-2.93049	-1.41532
50.82481		0,933
+18,63	+31,87	+13,22
-3,34	+1,14	+0,62
+3,13	+15,89	+7,19

2428120.5 = 1935 Nov. 14.0 UT

-2.7271130	-0.0454361	+0.2505845
7.5020023		0.0057604,933
-5.14664	-4.46558	-1.58657
48.94651		3,300
+6.36185	-3.05274	-1.38524
51.71125		0,909
+17,72	+32,38	+13,46
-3,36	+1,11	+0,61
+3,16	+15,98	+7,58

2428140.5 = 1935 Dec. 4.0 UT

-2.7263965	-0.2641153	+0.2407076
7.5609349		0.0056932,758
-5.01211	-4.73750	-1.62256
50.19785		3,178
+6.39517	-3.17433	-1.35638
52.81434		0,881
+16,79	+32.87	+13,69
-3,37	+1,08	+0,59
+3,13	+16,10	+7,93

2428160.5 = 1935 Dec. 24.0 UT

-2.7101675	-0.4812960	+0.2294602
7.6293057		0.0056169,161
-4.86029	-5.00449	-1.65849
51.41793		3,065
+6.44284	-3.29405	-1.32869
54.12637		0,849
+15,85	+33,34	+13,92
-3,39	+1,04	+0,58
+3,03	+16,24	+8,29

2428180.5 = 1936 Jan. 13.0 UT

-2.6787247	-0.6957794	+0.2169238
7.7067309		0.0055324,842
-4.69160	-5.26527	-1.69426
52.60470		2,962
+6.50454	-3.41070	-1.30209
55.67146		0,814
+14,88	+33,79	+14,13
-3,41	+1,01	+0,56
+2,87	+16,43	+8,61

2428200.5 = 1936 Feb. 2.0 UT

-2.6324700	-0.9064203	+0.2031868
7.7927809		0.0054411,007
-4.50654	-5.51868	-1.72973
53.75670		2,867
+6.57985	-3.52315	-1.27648
57.33641		0,779
+13,91	+34,22	+14,34
-3,42	+0,97	+0,55
+2,69	+16,62	+8,94

2428220.5 = 1936 Feb. 22.0 UT

-2.5718989	-1.1121369	+0.1883438
7.8869858		0.0053439,068
-4.30570	-5.76358	-1.76482
54.87250		2,780
+6.66828	-3.63034	-1.25179
59.21230		0,742
+12,91	+34,63	+14,54
-3,44	+0,94	+0,54
+2,45	+16,85	+9,24

2428240.5 = 1936 Mar. 13.0 UT

-2.4975901	-1.3119185	+0.1724936
7.9888405		0.0209681,359
-4.08977	-5.99890	-1.79938
55.95079		10,801
+6.76925	-3.73126	-1.22792
61.25283		2,822
+47,58	+140,06	+58,93
-13,82	+3,61	+2,09
+8,69	+68,35	+38,12

2428260.5 = 1936 Apr. 2.0 UT

-2.4101942	-1.5048314	+0.1557387
8.0978082		0.0051365,822
-3.85950	-6.22369	-1.83331
56.99108		2,627
+6.88210	-3.82499	-1.20476
63.44530		0,669
+10,87	+35,38	+14,91
-3,47	+0,87	+0,51
+1,87	+17,34	+9,80

2428280.5 =	1936 Apr. 22.0 UT	
-2.3104224	-1.6900234	+0.1381823
8.2133251		0.0201143,932
-3.61573	-6.43704	-1.86647
57.99270		10,236
+7.00611	-3.91069	-1.18223
65.77674		2,535
+39,28	+142,87	+60,33
-13,94	+3,32	+1,98
+6,09	+70,39	+40,21

2428300.5 =	1936 May 12.0 UT	
-2.1990358	-1.8667258	+0.1199303
8.3348069		0.0049190,605
-3.35932	-6.63818	-1.89875
58.95572		2,497
+7.14051	-3.98761	-1.16022
68.23403		0,600
+8,76	+36,03	+15,24
-3,50	+0,80	+0,48
+1,16	+17,86	+10,28

2428320.5 =	1936 June 1.0 UT	
-2.0768345	-2.0342544	+0.1010873
8.4616511		0.0192354691
-3.09115	-6.82634	-1.93002
59.87910		9,756
+7.28451	-4.05507	-1.13863
70.80416		2,271
+30,75	+145,29	+61,58
-14,06	+3,04	+1,86
+3,08	+72,52	+42,02

2428340.5 =	1936 June 21.0 UT	
-1.9446478	-2.1920093	+0.0817570
8.5932440		0.0046988,303
-2.81221	-7.00094	-1.96016
60.76391		2,386
+7.43724	-4.11250	-1.11736
73.47369		0,537
+ 6,60	+36,59	+15,53
-3,53	+0,72	+0,45
+0,25	+18,40	+10,70

2428360.5 =	1936 July 11.0 UT	
-1.8033244	-2.3394727	+0.0620412
8.7289605		0.0183586,889
-2.52343	-7.16140	-1.98907
61.60975		9,348
+7.59790	-4.15937	-1.09632
76.23036		2,033
+22,00	+147,31	+62,66
-14,17	+2,74	+1,75
-0,31	+74,65	+43,59

2428380.5 =	1936 July 31.0 UT	
-1.6537247	-2.4762070	+0.0420393
8.8681738		0.0044820,240
-2.22581	-7.30728	-2.01661
62.41729		2,292
+7.76558	-4.19527	-1.07541
79.06103		0,481
+4,39	+37,04	+15,78
-3,56	+0,65	+0,42
-0,54	+18,92	+11,06

2428400.5 =	1936 Aug. 20.0 UT	
-1.4967126	-2.6018509	+0.0218476
9.0102540		0.0175057,169
-1.92031	-7.43818	-2.04269
63.18669		9,000
+7.93942	-4.21985	-1.05454
81.95358		1,823
+13,04	+148,90	+63,56
-14,27	+2,45	+1,63
-4,04	+76,71	+44,88

2428420.5 =	1936 Sept. 9.0 UT	
-1.3331495	-2.7161106	+0.0015588
9.1545468		0.0042733,670
-1.60792	-7.55380	-2.06720
63.91862		2,212
+8.11857	-4.23283	-1.03363
84.89642		0,432
+2,12	+37,38	+15,99
-3,58	+0,58	+0,39
-1,51	+19,42	+11,36

2428440.5 =	1936 Sept. 29.0 UT	
-1.1638882	-2.8187806	-0.0187380
9.3005112		0.0166926476
-1.28961	-7.65390	-2.09005
64.61359		8,703
+8.30216	-4.23401	-1.01259
87.87804		1,620
+3,90	+150,05	+64,28
-14,37	+2,15	+1,51
-8,24	+78,73	+45,96

2428480.5 =	1936 Nov. 8.0 UT	
-0.8116116	-2.9887441	-0.0590211
9.5947882		0.0159306,084
-0.63904	-7.80692	-2.13036
65.89481		8,451
+8.67933	-4.20043	-0.96979
93.91487		1,487
-5,40	+150,74	+64,80
-14,46	+1,85	+1,39
-12,35	+80,37	+46,74

t	^{ii}f	^{i}f	f_x	Δ^{i}	Δ^{ii}	Δ^{iii}	Δ^{iv}	Δ^{v}	Δ^{vi}
7940	+2.04605707		-0.01194515						
		+0.13267853		-86281					
7960	+2.17873569		-0.01280796		+9260				
		+0.11987057		-77021		+1111			
7980	+2.29860617		-0.01357817		+10371		-188		
		+0.10629240		-66650		+ 923		-25	
8000	+2.40489857		-0.01424467	(-61003)	+11294	(+ 816)	-213	(-26)	
		+0.09204773		-55356		+ 710		-28	
8020	+2.49694630		-0.01479823		+12004		-241		
		+0.07724950		-43352		+ 469		-10	
8040	+2.57419580		-0.01523175		+12473		-251		
		+0.06201775		-30879		+ 218		- 1	
8060	+2.63621355		-0.01554054		+12691		-252		
		+0.04647721		-18188		- 34		+13	
8080	+2.68269076		-0.01572242		+12657		-239		
		+0.03075479		- 5531		- 273		+14	
8100	+2.71344555		-0.01577773		+12384		-225		
		+0.01497706		+ 6853		- 498		+31	
8120	+2.72842261		-0.01570920		+11886		-194		
		-0.00073214		+18739		- 692		+36	
8140	+2.72769047		-0.01552181		+11194		-168		
		-0.01625395		+29933		- 850		+42	
8160	+2.71143652		-0.01522248		+10344		-126		
		-0.03147643		+40277		- 976		+41	
8180	+2.67996009		-0.01481971		+9368		-85		
		-0.04629614		+49645		-1061		+38	
8200	+2.63366395		-0.01432326		+8307		-47		
		-0.06061940		+57952		-1108		+31	
8220	+2.57304455		-0.01374374		+7199		-16		
		-0.07436314		+65151		-1124		+33	
8240	+2.49868141		-0.01309223		+6075		+17		
		-0.08745537		+71226		-1107		+24	
8260	+2.41122604		-0.01237997		+4968		+41		
		-0.09983534		+76194		-1066		+20	
8280	+2.31139070		-0.01161803		+3902		+61		
		-0.11145337		+80096		-1005		+13	
8300	+2.19993733		-0.01081707		+2897		+74		
		-0.12227044		+82993		- 931		+19	
8320	+2.07766689		-0.00998714		+1966		+93		
		-0.13225758		+84959		- 838		-11	
8340	+1.94540931		-0.00913755		+1128		+82		
		-0.14139513		+86087		- 756		+16	
8360	+1.80401418		-0.00827668		+ 372		+98		
		-0.14967181		+86459		- 658		-10	
8380	+1.65434237		-0.00741209		- 286		+88		
		-0.15708390		+86173		- 570			
8400	+1.49725847		-0.00655036		- 856				
		-0.16363426		+85317					
8420	+1.33362421		-0.00569719						

t	^{ii}f	^{i}f	f_x	Δ^{i}	Δ^{ii}	Δ^{iii}	Δ^{iv}	Δ^{v}	Δ^{vi}
8240			-0.0523689						
				+58968					
8280	+2.3142976		-0.0464721		+6267				
		-0.2341327		+65235		-3083			
8320	+2.0801649		-0.0399486		+3184		+533		
		-0.2740813		+68419		-2550		+52	
8360	+1.8060836		-0.0331067		+ 634		+585		-78
		-0.3071880		+69053		-1965		-26	
8400	+1.4988956		-0.0262014		-1331		+559		-52
		-0.3333894		+67722		-1406		-78	
8440	+1.1655062		-0.0194292		-2737		+481		-13
		-0.3528186		+64985		- 925		-91	
8480	+0.8126876		-0.0129307		-3662		+390		- 1
		-0.3657493		+61323		- 535		-92	
8520	+0.4469383		-0.0067984		-4197		+298		
		-0.3725477		+57126		- 237			
8560	+0.0743906		-0.0010858		-4434				
		-0.3736335		+52692					
8600	-0.2992429		+0.0041834						
	+1.0		+0.0833333	+0.083333	+0.079167	+0.075	+0.07135	+0.0682	+0.065
	+1.0		+0.0833333		-0.004167	-0.004167	-0.003654	-0.00314	-0.0027

t	${}^{II}f$	${}^{I}f$	f_y	Δ^{i}	Δ^{ii}	Δ^{iii}	Δ^{iv}	Δ^{v}	Δ^{vi}
7940	-1.78358828		+0.01041445						
		+0.17608689		- 96289					
7960	-1.60750139		+0.00945156		-8727				
		+0.18553845		-105016		+1153			
7980	-1.42196294		+0.00840140		-7574		+170		
		+0.19393985		-112590		+1323		-35	
8000	-1.22802309		+0.00727550	(-115716)	-6251	(+1390)	+135	(-38)	
		+0.20121535		-118841		+1458		-41	
8020	-1.02680774		+0.00608709		-4793		+ 94		
		+0.20730244		-123634		+1552		-53	
8040	-0.81950530		+0.00485075		-3241		+ 41		
		+0.21215319		-126875		+1593		-46	
8060	-0.60735211		+0.00358200		-1648		- 5		
		+0.21573519		-128523		+1588		-48	
8080	-0.39161692		+0.00229677		- 60		- 53		
		+0.21803196		-128583		+1535		-42	
8100	-0.17358496		+0.00101094		+1475		-95		
		+0.21904290		-127108		+1440		-38	
8120	+0.04545794		-0.00026014		+2915		-133		
		+0.21878276		-124193		+1307		-22	
8140	+0.26424070		-0.00150207		+4222		-155		
		+0.21728069		-119971		+1152		-26	
8160	+0.48152139		-0.00270178		+5374		-181		
		+0.21457891		-114597		+ 971		- 3	
8180	+0.69610030		-0.00384775		+6345		-184		
		+0.21073116		-108252		+ 787		- 9	
8200	+0.90683146		-0.00493027		+7132		-193		
		+0.20580089		-101120		+ 594		+12	
8220	+1.11263235		-0.00594147		+7726		-181		
		+0.19985942		- 93394		+ 413		+11	
8240	+1.31249177		-0.00687541		+8139		-170		
		+0.19298401		- 85255		+ 243		+12	
8260	+1.50547578		-0.00772796		+8382		-158		
		+0.18525605		- 76873		+ 85		+22	
8280	+1.69073183		-0.00849669		+8467		-136		
		+0.17675936		- 68406		- 51		+22	
8300	+1.86749119		-0.00918075		+8416		-114		
		+0.16757861		- 59990		- 165		+16	
8320	+2.03506980		-0.00978065		+8251		- 98		
		+0.15779796		- 51739		- 263		+26	
8340	+2.19286776		-0.01029804		+7988		- 72		
		+0.14749992		- 43751		- 335		+15	
8360	+2.34036768		-0.01073555		+7653		- 57		
		+0.13676437		- 36098		- 392		+15	
8380	+2.47713205		-0.01109653		+7261		- 42		
		+0.12566784		- 28837		- 434			
8400	+2.60279989		-0.01138490		+6827				
		+0.11428294		- 22010					
8420	+2.71708283		-0.01160500						

t	${}^{II}f$	${}^{I}f$	f_y	Δ^{i}	Δ^{ii}	Δ^{iii}	Δ^{iv}	Δ^{v}	Δ^{vi}
8240			-0.0275017						
				-64851					
8280	+1.6928613		-0.0339868		+13493				
		+0.3446588		-51358		- 331			
8320	+2.0375201		-0.0391226		+13162		-609		
		+0.3055363		-38196		- 940		+247	
8360	+2.3430564		-0.0429422		+12222		-362		-44
		+0.2625941		-25974		-1302		+203	
8400	+2.6056505		-0.0455396		+10920		-159		-45
		+0.2170545		-15054		-1461		+158	
8440	+2.8227050		-0.0470450		+ 9459		- 1		-68
		+0.1700095		- 5595		-1462		+ 90	
8480	+2.9927145		-0.0476045		+ 7997		+ 89		-24
		+0.1224050		+ 2402		-1373		+ 66	
8520	+3.1151195		-0.0473643		+ 6624		+155		
		+0.0750407		+ 9026		-1218			
8560	+3.1901602		-0.0464617		+ 5406				
		+0.0285790		+14432					
8600	+3.2187392		-0.0450185						

These latter coordinates satisfy exactly the differential equation

$$\frac{d^2x_0}{dt^2} = -\frac{x_0}{r_0^3} \qquad ((7,3))$$

and similarly for y and z . Also let $x - x_0 = \xi$, $y - y_0 = \eta$, $z - z_0 = \zeta$. If we subtract ((7,3)) from ((3,3)), we shall have a second order differential equation for ξ, and similarly for η and ζ. The planetary terms are still exactly the same as before, but the principal term is now only the differential attraction of the Sun between the position where the planet actually is and where it would be

t	${}^{II}f$	${}^{I}f$	f_z	Δ^{I}	Δ^{II}	Δ^{III}	Δ^{IV}	Δ^{V}	Δ^{VI}
7940	-0.26952438		+0.00157389						
		-0.00439527		+ 3681					
7960	-0.27391965		+0.00161070		-1250				
		-0.00278457		+ 2431		- 51			
7980	-0.27670422		+0.00163501		-1301		+25		
		-0.00114956		+ 1130		- 26		+1	
8000	-0.27785378		+0.00164631	(+ 466)	-1327	(- 13)	+26	(0)	
		+0.00049675		- 197		0		0	
8020	-0.27735703		+0.00164434		-1327		+26		
		+0.00214109		- 1524		+ 26		-2	
8040	-0.27521594		+0.00162910		-1301		+24		
		+0.00377019		- 2825		+ 50		+2	
8060	-0.27144575		+0.00160085		-1251		+26		
		+0.00537104		- 4076		+ 76		-9	
8080	-0.26607471		+0.00156009		-1175		+17		
		+0.00693113		- 5251		+ 93		+1	
8100	-0.25914358		+0.00150758		-1082		+18		
		+0.00843871		- 6333		+111		-6	
8120	-0.25070487		+0.00144425		- 971		+12		
		+0.00988296		- 7304		+123		-6	
8140	-0.24082191		+0.00137121		- 848		+ 7		
		+0.01125417		- 8152		+130		-3	
8160	-0.22956774		+0.00128969		- 718		+ 4		
		+0.01254386		- 8870		+134		-3	
8180	-0.21702388		+0.00120099		- 584		+ 1		
		+0.01374485		- 9454		+135		-7	
8200	-0.20327903		+0.00110645		- 449		- 8		
		+0.01485130		- 9903		+127		+5	
8220	-0.18842773		+0.00100742		- 322		- 3		
		+0.01585872		-10225		+124		-7	
8240	-0.17256901		+0.00090517		- 198		-10		
		+0.01676389		-10423		+114		-3	
8260	-0.15580512		+0.00080094		- 84		-13		
		+0.01756483		-10507		+101		+5	
8280	-0.13824029		+0.00069587		+ 17		- 8		
		+0.01826070		-10490		+ 93		-9	
8300	-0.11997959		+0.00059097		+ 110		-17		
		+0.01885167		-10380		+ 76		+10	
8320	-0.10112792		+0.00048717		+ 186		- 7		
		+0.01933884		-10194		+ 69		-9	
8340	-0.08178908		+0.00038523		+ 255		-16		
		+0.01972407		- 9939		+ 53		+7	
8360	-0.06206501		+0.00028584		+ 308		- 9		
		+0.02000991		- 9631		+ 44		-2	
8380	-0.04205510		+0.00018953		+ 352		-11		
		+0.02019944		- 9279		+ 33			
8400	-0.02185566		+0.00009674		+ 385				
		+0.02029618		- 8894					
8420	-0.00155948		+0.00000780						
8240			+0.0036207						
				-8372					
8280	-0.1384142		+0.0027835		+ 24				
		+0.0371646		-8348		+271			
8320	-0.1012496		+0.0019487		+295		-78		
		+0.0391133		-8053		+193		+11	
8360	-0.0621363		+0.0011434		+488		-67		
		+0.0402567		-7565		+126		+ 4	
8400	-0.0218796		+0.0003869		+614		-63		
		+0.0406436		-6951		+ 63		+21	
8440	+0.0187640		-0.0003082		+677		-42		
		+0.0403354		-6274		+ 21		+ 9	
8480	+0.0590994		-0.0009356		+698		-33		
		+0.0393998		-5576		-12		+12	
8520	+0.0984992		-0.0014932		+686		-21		
		+0.0379066		-4890		-33			
8560	+0.1364058		-0.0019822		+653				
		+0.0359244		-4237					
8600	+0.1723302		-0.0024059						

in its elliptic orbit. It would achieve no good purpose, if one is intent upon reducing the number of significant figures which must be computed, to compute these attractions separately and difference them. Therefore Encke transforms the expression as follows. Let $h = (wk)^2/r_0^3$, and write the result of ((3,3)) minus ((7,3)) as

$$\frac{d^2\xi}{dt^2} = h\left[\left(1 - \frac{r_0^3}{r^3}\right)x - \xi\right] + P_x = hfqx - h\xi + P_x \qquad ((7,4))$$

which is a definition of fq that will be evaluated below. By substitution of $x = x_0 + \xi$, etc. into

r^2, we have
$$\frac{r^2}{r_0^2} = 1 + 2\,\frac{(x_0+\frac{1}{2}\xi)\xi + (y_0+\frac{1}{2}\eta)\eta + (z_0+\frac{1}{2}\zeta)\zeta}{r_0^2} = 1 + 2q \qquad ((7,5))$$

which defines q. Then $fq = 1 - (1+2q)^{-3/2}$, so that f is a function of q. By expanding the right side into a series, we obtain
$$f = 3\left[1 - \frac{5}{2}q + \frac{5\ 7}{2\ 3}q^2 - \frac{5\ 7\ 9}{2\ 3\ 4}q^3 + \ldots\right] \qquad ((7,6))$$

but it is not necessary to use this expression, as f is tabulated as a function of q among the tables at the end of the book, and it is tabulated more extensively as Table XI in Planetary Coordinates. The example which follows will illustrate the application of these equations to the same problem we have just considered on the previous pages.

The principal advantage of Encke's method arises from the fact that when the actual deviation from the elliptic path is small, ξ, η, and ζ are small numbers and the integrations may be computed with longer intervals than are possible when the coordinates are integrated directly, and the solar term enters with its full value. Counteracting this is the fact, which can be observed by comparing the examples, that it requires more work to compute a single step in Encke's method. However, the size of the perturbations and other things being about equal, Encke's method does not suffer from the effects of a highly eccentric orbit as the direct integration of the coordinates does. It does suffer, eventually, from a cause which is entirely absent in the other method, namely the need for "rectification". As the object continues to deviate more and more from its elliptical path, the perturbations increase without limit, so that the integration tables become unmanageable. It is then necessary to adopt some date for which $x = x_0 + \xi$, $x' = x_0' + \xi'$, etc. (in units of 1/k mean solar days) may be reliably obtained and determine a new osculating orbit by means of the equations on page 47 to serve as the reference ellipse. Then the integrations are commenced again, with both $\xi = 0$ and $\xi' = 0$ at the chosen epoch. Since the direct integration of the coordinates is continuously osculating at every date, such a situation cannot arise in that method.

We shall now use Encke's method to compute the perturbations for (1361) from 1935 to 1939, using an 80 day interval and JD 2428000.5 as the epoch. In the block of computation for each date, the planetary terms correspond to an interval of 40 days, and the combined result is multiplied by 4 just before writing P. The solar terms are given for an 80 day interval. The quantities in the integration tables shown in ( ) were used to interpolate the perturbations shown on page 85 by means of ((1,19)). Each of these examples by the two different methods was computed independently at different times and no attempt has been made to bring them into closer agreement for the sake of nicety of appearances.

The method of integrating directly for the heliocentric rectangular coordinates is usually referred to as "Cowell's method". This is not strictly accurate, as the reader may verify by an examination of the Appendix to the Greenwich Observations of 1909, where the return of Halley's Comet computed by Cowell and Crommelin is published. What Cowell did was to take the second difference of equation ((1,10)); thus
$$\Delta^{ii} x_i = f_i + \frac{1}{12}\Delta_i^{ii} - \frac{1}{240}\Delta_i^{iv} + \ldots \qquad ((7,7))$$

At the conclusion of the paper he recommends against this practice and in favor of the same one we have used above. For work done with hand calculating machines it is still best to follow these recommendations today, but the reader will find in the Astronomical Journal, v. 52, p. 115, an exposition of a successful procedure for computing with electric punched card machines which is based upon the original Cowell's method. Due to the extensive literature in which use has been made of the term "Cowell's method", it now seems impossible to maintain a distinction between the two variations of this process.

The first problem to which Cowell applied his method was the orbit of the Eighth Satellite of Jupiter. Since all the satellite orbits which are not close in to the primary are considerably affected by solar perturbations, they must be treated as a three body problem. As previously intimated, the generality of the La Placian method as presented in Chapter 4 permits the solution for a preliminary disturbed orbit with no change in the fundamental principles. We shall now show how such a preliminary disturbed orbit may be derived, and how the trajectory is computed by Cowell's method, using illustrations from the author's initial computations of Jupiter XI.

Date	M	E	1935 II 7	328.17022	324.08535
cos E	cos E - e	sin E	+0.8098917	+0.6883490	-0.5865794
r_0^2		h	7.7506121		0.0877691
$-x_0$	$-y_0$	$-z_0$	-1.2226985	+2.4909850	+0.2249760
$-\xi$	$-\eta$	$-\zeta$	+29	-539	-123
-x	-y	-z	-1.2226956	+2.4909311	+0.2249637
q	f	-hfq	-181,374	3.000136	+47,759
$(x_j - x)$	$(y_j - y)$	$(z_j - z)$	-5.28005	-0.85431	-1.11096
ρ_j^2		$m_j (wk)^2/\rho_j$	29.84300		27,728
$(x_s - x)$	$(y_s - y)$	$(z_s - z)$	+7.28941	-1.82917	-1.92873
ρ_s^2		$m_s (wk)^2/\rho_s$	60.20136		2,896
X (Jup)	Y (Jup)	Z (Jup)	+114,84	+94,68	+37,81
X (Sat)	Y (Sat)	Z (Sat)	-12,29	+6,24	+3,11
P_x	P_y	P_z	-90,98	+287,74	+18,12

1935 IV 28	342.70097	340.36039	1935 VII 17	357.23171	356.84891
+0.9418253	+0.8202826	-0.3361028	+0.9984880	+0.8769453	-0.0549691
7.4773534		0.0926240	7.3614989		0.0948191
-1.9005146	+1.9483330	+0.2634316	-2.4037101	+1.2274161	+0.2777164
-5	-130	-36	0	0	0
-1.9005151	+1.9483200	+0.2634280			
-33,880	3.000025	+9,412			
-5.54105	-1.76215	-1.23938	-5.58392	-2.80391	-1.37401
35.34447		21,513	40.92998		17,263
+6.79878	-2.00636	-1.74726	+6.46509	-2.35375	-1.58587
53.30181		3,476	49.85251		3,843
+103,87	+105,87	+42,88	+91,61	+116,13	+47,58
-12,64	+5,74	+2,92	-12,97	+5,24	+2,72
-17,37	+266,91	+52,26	+28,36	+255,68	+81,94

1935 X 5	11.76245	13.37314	1935 XII 24	26.29320	29.74865
+0.9728844	+0.8513417	+0.2312918	+0.8682105	+0.7466678	+0.4961960
7.4137369		0.0938188	7.6292184		0.0898722
-2.6813773	+0.3914380	+0.2659491	-2.7101584	-0.4812473	+0.2294795
-18	-126	-45	-83	-494	-193
-2.6813791	+0.3914254	+0.2659446	-2.7101667	-0.4812967	+0.2294602
-1,757	3.000001	+0,495	+54,842	2.999959	-14,786
-5.36290	-3.91208	-1.51476	-4.86029	-5.00449	-1.65849
46.35956		14,321	51.41793		12,261
+6.33882	-2.80885	-1.44670	+6.44284	-3.29405	-1.32869
50.16322		3,807	54.12637		3,397
+78,10	+125,34	+51,86	+63,39	+133,36	+55,66
-13,27	+4,71	+2,52	-13,56	+4,17	+2,31
+48,64	+253,33	+108,72	+48,50	+259,92	+132,49

1936 III 13	40.82394	45.81794	1936 VI 1	55.35469	61.47311
+0.6969406	+0.5753979	+0.7171289	+0.4775712	+0.3560285	+0.8785931
7.9884523		0.0838787	8.4606441		0.0769556
-2.4975660	-1.3118105	+0.1725387	-2.0767783	-2.0340602	+0.1011687
-235	-1088	-453	-554	-1956	-816
-2.4975895	-1.3119193	+0.1724934	-2.0768337	-2.0342558	+0.1010871
+242,361	2.999818	-60,983	+596,508	2.999553	-137,693
-4.08977	-5.99890	-1.79938	-3.09115	-6.82635	-1.93002
55.95079		10,801	59.87924		9,756
+6.76925	-3.73126	-1.22792	+7.28451	-4.05508	-1.13863
61.25283		2,822	70.80424		2,271
+47,58	+140,06	+58,93	+30,75	+145,29	+61,58
-13,82	+3,61	+2,09	-14,06	+3,04	+1,86
+34,76	+273,38	+152,48	+12,30	+290,09	+168,10

	$''\xi$	$'\xi$								
1935 II 7	-0.0000017		-147							
		+25		+111						
IV 28	+0.0000008		-36		-47					
		-11		+64		+1				
VII 17	-0.0000003	(+3)	+28	(+41)	-46	(+32)	+62			
		+17		+18		+63		-91		
X 5	+0.0000014	(+40)	+46	(+26)	+17	(+48)	-29	(-72)	+38	
		+63		+35		+34		-53		
XII 24	+0.0000077		+81		+51		-82		+93	
		+144		+86		-48		+40		
1936 III 13	+0.0000221		+167		+3		-42		+27	
		+311		+89		-90		+67		
VI 1	+0.0000532		+256		-87		+25		-51	
		+567		+2		-65		+16		
VIII 20	+0.0001099		+258		-152		+41		-20	
		+825		-150		-24		-4		
XI 8	+0.0001924		+108		-176		+37		-9	
		+933		-326		+13		-13		
1937 I 27	+0.0002857	(+824)	-218	(-408)	-163	(+25)	+24	(-14)	-2	
		+715		-489		+37		-15		
IV 17	+0.0003572		-707		-126		+9		+15	
		+8		-615		+46		0		
VII 6	+0.0003580		-1322		-80		+9		-2	
		-1314		-695		+55		-2		
IX 24	+0.0002266		-2017		-25		+7		+17	
		-3331		-720		+62		+15		
XII 13	-0.0001065		-2737		+37		+22		-13	
		-6068		-683		+84		+2		
1938 III 3	-0.0007133	(-7778)	-3420	(-622)	+121	(+96)	+24	(+8)	+13	
		-9488		-562		+108		+15		
V 22	-0.0016621		-3982		+229		+39		-5	
		-13470		-333		+147		+10		
VIII 10	-0.0030091		-4315		+376		+49		-21	
		-17785		+43		+196		-11		
X 29	-0.0047876		-4272		+572		+38		-23	
		-22057		+615		+234		-34		
1939 I 17	-0.0069933		-3657		+806		+4		-71	
		-25714		+1421		+238		-105		
IV 7	-0.0095647	(-26832)	-2236	(+1943)	+1044	(+188)	-101			
		-27950		+2465		+137				
VI 26	-0.0123597		+229		+1181					
		-27721		+3646						
IX 14	-0.0151318		+3875							

1936 VIII 20	69.88543	76.66145	1936 XI 8	84.41617	91.37804
+0.2307044	+0.1091617	+0.9730039	-0.0240489	-0.1455916	+0.9997108
9.0082358		0.0700464	9.5913226		0.0637570
-1.4965997	-2.6015269	+0.0219745	-0.8114177	-2.9882205	-0.0588401
-1121	-3253	-1270	-1934	-5247	-1811
-1.4967118	-2.6018522	+0.0218475	-0.8116111	-2.9887452	-0.0590212
+1122,664	2.999158	-235,850	+1809,632	2.998644	-345,974
-1.92031	-7.43818	-2.04269	-0.63904	-7.80693	-2.13036
63.18669		9,000	65.89496		8,451
+7.93942	-4.21985	-1.05454	+8.67933	-4.20044	-0.96979
81.95358		1,823	93.91496		1,487
+13,04	+148,90	+63,56	-5,40	+150,74	+64,80
-14,27	+2,45	+1,63	-14,46	+1,85	+1,39
-16,16	+306,86	+179,53	-49,42	+321,47	+186,98

	$^{II}\eta$	$^{I}\eta$							
1935 II 7	+0.0000510		+359						
		-402		-86					
IV 28	+0.0000108		+273		+70				
		-129		-16		-68			
VII 17	-0.0000021	(-1)	+256	(-15)	+2	(-38)	+61		
		+127		-14		-7		+9	
X 5	+0.0000106	(+248)	+242	(-16)	-5	(+28)	+70	(-32)	-81
		+369		-19		+63		-72	
XII 24	+0.0000475		+223		+58		-2		+9
		+592		+39		+61		-63	
1936 III 13	+0.0001067		+262		+119		-65		+72
		+854		+158		-4		+9	
VI 1	+0.0001921		+420		+115		-56		+26
		+1274		+273		-60		+35	
VIII 20	+0.0003195		+693		+55		-21		-8
		+1967		+328		-81		+27	
XI 8	+0.0005162		+1021		-26		+6		-19
		+2988		+302		-75		+8	
1937 I 27	+0.0008150	(+3650)	+1323	(+252)	-101	(-68)	+14	(+5)	-6
		+4311		+201		-61		+2	
IV 17	+0.0012461		+1524		-162		+16		-12
		+5835		+39		-44		-10	
VII 6	+0.0018296		+1563		-207		+6		+11
		+7398		-168		-38		+1	
IX 24	+0.0025694		+1395		-245		+7		-8
		+8793		-413		-31		-7	
XII 13	+0.0034487		+982		-276		0		+22
		+9775		-689		-31		+15	
1938 III 3	+0.0044262	(+9922)	+293	(-842)	-307	(-24)	+15	(+10)	-9
		+10068		-996		-16		+6	
V 22	+0.0054330		-703		-323		+21		+29
		+9365		-1319		+5		+35	
VIII 10	+0.0063695		-2022		-318		+56		+8
		+7343		-1637		+61		+43	
X 29	+0.0071038		-3659		-257		+99		+30
		+3684		-1894		+160		+73	
1939 I 17	+0.0074722		-5553		-97		+172		-6
		-1869		-1991		+332		+67	
IV 17	+0.0072853	(-5641)	-7544	(-1874)	+235	(+452)	+239		
		-9413		-1756		+571			
VI 26	+0.0063440		-9300		+806				
		-18713		-950					
IX 14	+0.0044727		-10250						

1937 I 27	98.94692	105.65255
-0.2698031	-0.3913458	+0.9629155
10.1711384		0.0583836
-0.0740170	-3.1854615	-0.1359961
-2840	-8261	-2444
-0.0743010	-3.1862876	-0.1362405
+2640,982	2.998021	-462,265
+0.69214	-7.92318	-2.18723
68.03981		8,055
+9.45116	-3.98904	-0.87941
106.01023		1,240
-24,38	+150,68	+65,24
-14,62	+1,23	+1,14
-86,82	+332,57	+190,68

1937 IV 17	113.47766	119.53653
-0.4929783	-0.6145210	+0.8700416
10.7124248		0.0540149
+0.6682926	-3.1974462	-0.2052989
-3514	-12589	-3189
+0.6679412	-3.1987051	-0.2056178
+3600,307	2.997302	-582,886
+2.01802	-7.79119	-2.20894
69.65446		7,776
+10.20744	-3.59074	-0.77958
117.69299		1,059
-43,68	+148,58	+64,81
-14,75	+0,61	+0,89
-127,71	+339,21	+190,79

	$''\zeta$	$'\zeta$							
1935 II 7	+0.0000122		+18						
		-90		+33					
IV 28	+0.0000032		+51		-2				
		-39		+31		-6			
VII 17	-0.0000007	(+2)	+82	(+27)	-8	(-7)	-2		
		+43		+23		-8		+11	
X 5	+0.0000036	(+96)	+105	(+15)	-16	(-4)	+9		-11
		+148		+7		+1		0	
XII 24	+0.0000184		+112		-15		+9		-7
		+260		-8		+10		-7	
1936 III 13	+0.0000444		+104		-5		+2		-1
		+364		-13		+12		-8	
VI 1	+0.0000808		+91		+7		-6		+7
		+455		-6		+6		-1	
VIII 20	+0.0001263		+85		+13		-7		+5
		+540		+7		-1		+4	
XI 8	+0.0001803		+92		+12		-3		-2
		+632		+19		-4		+2	
1937 I 27	+0.0002435	(+688)	+111	(+23)	+8	(-4)	-1		-3
		+743		+27		-5		-1	
IV 17	+0.0003178		+138		+3		-2		+4
		+881		+30		-7		+3	
VII 6	+0.0004059		+168		-4		+1		-6
		+1049		+26		-6		-3	
IX 24	+0.0005108		+194		-10		-2		+6
		+1243		+16		-8		+3	
XII 13	+0.0006351		+210		-18		+1		-7
		+1453		-2		-7		-4	
1938 III 3	+0.0007804	(+1557)	+208	(-14)	-25	(-8)	-3		+4
		+1661		-27		-10		0	
V 22	+0.0009465		+181		-35		-3		+6
		+1842		-62		-13		+6	
VIII 10	+0.0011307		+119		-48		+3		-10
		+1961		-110		-10		-4	
X 29	+0.0013268		+9		-58		-1		+13
		+1970		-168		-11		+9	
1939 I 17	+0.0015238		-159		-69		+8		+6
		+1811		-237		-3		+15	
IV 17	+0.0017049	(+1613)	-396	(-273)	-72	(+8)	+23		
		+1415		-309		+20			
VI 26	+0.0018464		-705		-52				
		+710		-361					
IX 14	+0.0019174		-1066						

1937 VII 6	128.00841	133.09370
-0.6831935	-0.8047362	+0.7302374
11.1848471		0.0506291
+1.3750825	-3.0371890	-0.2635871
-3470	-18427	-4073
+1.3747355	-3.0390317	-0.2639944
+4674,784	2.996498	-709,212
+3.29015	-7.42487	-2.19249
70.76079		7,594
+10.90761	-3.01945	-0.66748
128.53856		0,928
-63,03	+144,33	+63,46
-14,85	-0,03	+0,63
-171,09	+340,45	+187,28

1937 IX 24	142.53915	146.39355
-0.8328590	-0.9544017	+0.5534852
11.5637247		0.0481614
+2.0127696	-2.7234637	-0.3085915
-2098	-25811	-5124
+2.0125598	-2.7260448	-0.3091039
+5853,529	2.995616	-844,506
+4.46687	-6.84480	-2.13619
71.36752		7,498
+11.51794	-2.29485	-0.54121
138.22219		0,832
-82,14	+137,84	+61,15
-14,91	-0,68	+0,36
-215,90	+335,71	+180,17

Let

$x,\ y,\ z,\ r$	be the Jovicentric	coordinates of the satellite
$\xi,\ \eta,\ \zeta,\ \rho$	be the geocentric	coordinates of the satellite
$[x], [y], [z], [r]$	be the heliocentric	coordinates of the satellite
$(\xi), (\eta), (\zeta), (\rho)$	be the geocentric	coordinates of Jupiter
$(x), (y), (z), (r)$	be the heliocentric	coordinates of Jupiter
$X,\ Y,\ Z,\ R$	be the geocentric	coordinates of the Sun.

Then $\xi = (\xi) + x$, $[x] = (x) + x$, $(\xi) = (x) + X$, etc. and the dynamical conditions upon the motion of the object are given by

$$\frac{d^2x}{dt^2} = -\frac{x}{r^3} + M\frac{(x)}{(r)^3} - M\frac{[x]}{[r]^3} \tag{(7,8)}$$

and similarly for y and z.

In the Jupiter satellite system, if Jupiter is taken as the unit of mass, then the mass of the Sun is $M = 1047.355$. The quantity in this system corresponding to the Gaussian constant in the solar system is $(k) = k\ \ 1/M$. It just so happens that if one changes the unit of length from one astronomical unit to 0.1 A.U., then (k) must be multiplied by $10^{3/2}$, i.e. $((k)) = 0.97713157k$. Thus one has a miniature system, one tenth the size of the solar system, in which Jupiter VI, VII, and X have orbits comparable to the position of Venus, and Jupiter VIII, IX, and XI correspond to Mars. In this way the magnitudes of the position and velocity vectors are both kept nearly to the order of unity. In the computations shown in the illustration, this latter system of units was not used.

To obtain a preliminary solution, let

$$U = \frac{y + (\eta)}{x + (\xi)} = \tan\alpha, \quad V = \frac{z + (\zeta)}{x + (\xi)} = \sec\alpha\tan\delta, \quad P = (\eta) - U(\xi), \quad Q = (\zeta) - V(\xi).$$

$$\begin{aligned} y &= Ux - P & z &= Vx - Q \\ y' &= U'x + Ux' - P' & z' &= V'x + Vx' - Q' \\ y'' &= U''x + 2U'x' + Ux'' - P'' & z'' &= V''x + 2V'x' + Vx'' - Q'' \end{aligned} \tag{(7,9)}$$

By substitution for x'', y'', and z'' from ((7,8)), the last two equations of ((7,9)) reduce to

$$\begin{aligned} D\,x = {} & (\tfrac{1}{2}P''V' - \tfrac{1}{2}Q''U') + M[(VU' - UV')(x) + V'(y) - U'(z)]/2(r)^3 + (PV' - QU')/2r^3 \\ & + \{-M[(VU' - UV')(x) + V'(y) - U'(z)] \quad + M(PV' - QU')\}/2[r]^3 \end{aligned}$$

$$\begin{aligned} D\,x' = {} & (\tfrac{1}{2}U''\tfrac{1}{2}Q'' - \tfrac{1}{2}V''\tfrac{1}{2}P'') - M[(\tfrac{1}{2}U''V - \tfrac{1}{2}V''U)(x) + \tfrac{1}{2}V''(y) - \tfrac{1}{2}U''(z)]/2(r)^3 + (\tfrac{1}{2}U''Q - \tfrac{1}{2}V''P)/2r^3 \\ & + \{+M[(\tfrac{1}{2}U''V - \tfrac{1}{2}V''U)(x) + \tfrac{1}{2}V''(y) - \tfrac{1}{2}U''(z)] \quad + M(\tfrac{1}{2}U''Q - \tfrac{1}{2}V''P)\}/2[r]^3 \end{aligned}$$

where $D = \frac{1}{2}U''V' - \frac{1}{2}V''U'$. Also

$$r^2 = (1 + U^2 + V^2)x^2 - 2(UP + VQ)x + (P^2 + Q^2)$$

$$[r]^2 = r^2 + 2\{(x) + U(y) + V(z)\}x - 2\{P(y) + Q(z)\} + (r)^2$$

Now the first equation may be solved simultaneously with the last two, and then x' is given by the second equation. In order to effect the solution, write the first equation as

$$\Delta(x) = x + C_0 + C_1/r^3 + C_2/[r]^3 = 0 \tag{(7,10)}$$

Solve by trials for such values of x as will give $\Delta(x) = 0$. In this problem there are two solutions to be expected, one in case the motion of the satellite about Jupiter is direct and the other in case the satellite is on the other side of Jupiter and moving in a retrograde orbit about Jupiter.

All the derivatives at the epoch are evaluated by means of the observations and Taylor's series, similar to ((4,14)), in the form:

$$\frac{W - W_0}{T} = W_0' + \tfrac{1}{2}W_0''T$$

where now $T = (k)(t - t_0)$. The following computations of the preliminary orbit of Jupiter XI need no further explanation. This orbit was corrected by the method of equations ((6,8)), using the closed expressions for df and dg, and then the path was projected forward by numerical integration. A sample of these computations is also shown.

In equation ((7,8)), the direct and indirect terms due to the Sun are of the same order of magnitude and of opposite sign, so that to a large extent they cancel each other. Therefore, it is pos-

	1938 July 30.4153 UT	Aug. 25.2813 UT	Oct. 2.2828 UT	
	JD 2429109.9153	135.7812	173.7828	
	(.8915)	(.7575)	(.7578)	
T	-0.01374876	0.03394812	+0.02019936	
α	22 16 58.23	22 04 16.67	21 47 49.01	
δ	-12 06 33.1	-13 29 24.5	-15 03 38.2	
U	-0.4825014	-0.5527187	-0.6505445	
V	-0.2382179	-0.2741026	-0.3210118	
P	-0.1251960	-0.1198233	-0.1015359	
Q	-0.0412187	-0.0290657	-0.0090591	
(x)	+4.146467	+4.251322	+4.393265	
(y)	-2.579505	-2.420076	-2.179054	
(z)	-1.208117	-1.142288	-1.042358	
(ξ)	+3.539366	+3.362596	+3.403775	
(η)	-1.832945	-1.978393	-2.315843	
(ζ)	-0.884359	-0.950762	-1.101711	
	(W,1)	(W,3)	W_0'	$\frac{1}{2}W_0''$
U	-5.107173	-4.843015	-5.000191	+7.781226
V	-2.610031	-2.322311	-2.493506	+8.475285
P	+0.390777	+0.905346	+0.599174	+15.157511
Q	+0.883934	+0.990457	+0.927075	+3.137817

$$D = +22.97551$$

$$\Delta(x) = 0 = +0.003339340/r^3 - 3.1199112/[r]^3 - 0.9099398 - x_0$$

$$r^2 = +1.3806302\,x^2 - 0.1483911\,x + 0.0152024$$

$$[r]^2 = +1.3806302\,x^2 + 11.6557037\,x + 24.604165$$

	Direct Motion			Retrograde Motion	
x	-0.0428018	x'	+0.2098242	+0.1393066	+1.1300326
y	+0.1434807	y'	-0.5011306	+0.0428259	-1.9203238
z	+0.0407978	z'	-0.8778618	-0.0091186	-1.5841817
r^2	0.02408317	G^2	1.0657994	0.02132353	7.4742488
r	0.1551875			0.1460258	
a	0.0845892	P	0.02460212	0.1607210	0.06443308
$\sqrt{a}$	0.2908423	n		0.4009002	0.008249486
e sin E	-0.4012424			+0.2235642	0.05834098
e cos E	-0.8346013	e	0.9260425	+0.0914332	0.2415388
1 - e cos E	1.8346013			+0.9085668	13°83915

The direct orbit was discarded because of the unreasonably large eccentricity.

---

sible to perform the computations in a manner similar to ((7,4)) in Encke's method. In this case, [x], (x), and x take the place of x, x_0, and ξ. Let

$$q = \frac{\frac{1}{2}r^2 + (x)x + (y)y + (z)z}{(r)^2}, \quad (h) = \frac{(wk)^2}{(r)^3}, \quad h = m_j\frac{(wk)^2}{r^3} \qquad ((7,11))$$

Then ((7,8)) becomes

$$\frac{d^2x}{dt^2} = [(h)fq - (h) - h]x + (h)fq\,(x) \qquad ((7,12))$$

and it is this equation which was used in the computation of the example. If q exceeds the limits of the tables, we may use

$$fq = 1 - 31.6227767\,(10 + 20q)^{-3/2}$$

It is very important that the positions and coordinates of Jupiter and the satellite shall be referred strictly to the same coordinate system; otherwise any systematic difference between the two will become a part (a spurious part) of the satellite's residual. The author recommends the following method of computation of the satellite's residuals. Correct the tabular ephemeris of Jupiter printed in the annual ephemerides so that it agrees with the observed correction of Jupiter in the star system that is used for the satellite positions. Remove the part of the "apparent place reduction" which is due to precession and nutation, and then apply the precession to 1950.0. Then

$$(\xi) = (\rho)\cos\delta\cos\alpha, \qquad (\eta) = (\rho)\cos\delta\sin\alpha, \qquad (\zeta) = (\rho)\sin\delta$$

are the geometrical coordinates of Jupiter at the time, $t(\text{obs}) - 0.00577\,(\rho) = t_j$.

Let $t(\text{obs}) - 0.00577\rho = t_s$, the time at which the light left the satellite, and the time for which we compute x, y, z. Then to (ξ) we must add $(x)' =$ (motion of Jupiter in x per day) $(t_s - t_j)$, and similarly for y and z. Then

$$\begin{aligned} \Delta X + (\xi) + (x)' + x &= \rho\cos\delta\cos\alpha \\ \Delta Y + (\eta) + (y)' + y &= \rho\cos\delta\sin\alpha \\ \Delta Z + (\zeta) + (z)' + z &= \rho\sin\delta \end{aligned} \qquad ((7,13))$$

where ΔX, etc. are the topocentric corrections, and the computed values of α and δ on the right are to be compared with the observed values.

Thus far in this chapter, we have not considered the process of integrating the equations ((3,2)) directly, in other words, the use of barycentric coordinates, in which the center of gravity of all the bodies is taken as the origin. This has the obvious advantage that there are no indirect terms to be computed, but it also has several practical disadvantages which were discussed by Comrie in the M. N. R. A. S. at the time when the publication of Planetary Coordinates was contemplated. The chief objection is that the coordinates of each of the perturbing bodies changes whenever one of the other perturbing bodies is included or excluded from the computations, because this changes the barycenter. But this does not exclude the use of this method with profit in any special problem, such as the return of a long period comet. Another example of this process will be found in the Astronomical Journal, v. 47, p. 17. Here it is the barycenter of only three inner planets which was used, so as to eliminate the indirect term of Venus, which alone was the cause of requiring a small interval. Thus a larger interval became permissible.

There is another entirely different approach to this problem of computing the actual trajectory of an object in the solar system, and it is based upon the seemingly paradoxical terminology known as the Method of Variation of Arbitrary Constants. This is usually referred to as the perturbations in the elements. In this case, the Arbitrary Constants are the usual elements of the orbit, i, Ω, ω, a, e, T, which are the constants of integration of the Two Body Problem. Then the perturbation problem is to compute the way in which these elements or "constants" should be varied as a function of the time so that the position in space computed from the set of elements corresponding to any given instant of time will be the same as its true position at that time. This leads to six single integrals, and a complete exposition of the method is given by Stracke: Bahnbestimmung der Planeten und Kometen, p. 284, but the computational work is much more complex and less routine than the methods given above.

Hansen devised a method which is intermediate between the perturbations in the elements and in the rectangular coordinates. In effect, Hansen computes three components of the perturbations referred to a fixed elliptic orbit (as in Encke's method) but they are computed essentially with respect to a rotating coordinate system which moves with the object in its orbit. The equations for this method will be found in Watson: Theoretical Astronomy, ((110)), ((115)), ((129)), and in Astronomische Nachrichten 799 and 882. Even this method, however, can not be computed with the same facility as the rectangular coordinate methods. In this case the disadvantage is due mainly to the extra work required by the moving coordinate system.

Long years of experience with the computation of minor planet orbits at the Rechen Institut led G. Stracke to develop a method of approximate perturbation computations which is adequate for the preparation of search ephemerides for long periods of time. This method was published in the Veroffentlichungen des Rechen-Instituts, No. 48. It is a simplified form of the equations for

Date	^{II}x	^{I}x	f_x	Δ			
1938 VII 11	+0.09800823		-0.00121710				
		+0.01095083		-4975			
VII 21	+0.10895906		-0.00126685		+2573		
		+0.00968398		-2402		-404	
VII 31	+0.11864304		-0.00129087		+2169		+23
		+0.00839311		- 233		-381	
VIII 10	+0.12703615		-0.00129320		+1788		+39
		+0.00709991		+1555		-342	
VIII 20	+0.13413606		-0.00127765		+1446		+36
		+0.00582226		+3001		-306	
VIII 30	+0.13995832		-0.00124764		+1140		+43
		+0.00457462		+4141		-263	
IX 9	+0.14453294		-0.00120623		+877		+33
		+0.00336839		+5018		-230	
IX 19	+0.14790133		-0.00115605		+647		+33
		+0.00221234		+5665		-197	
IX 29	+0.15011367		-0.00109940		+450		+32
		+0.00111294		+6115		-165	
X 9	+0.15122661		-0.00103825		+285		+24
		+0.00007469		+6400		-141	
X 19	+0.15130130		-0.00097425		+144		+23
		-0.00089956		+6544		-118	
X 29	+0.15040174		-0.00090881		+26		+25
		-0.00180837		+6570		-93	
XI 8	+0.14859337		-0.00084311		-67		+16
		-0.00265148		+6503		-77	
XI 18	+0.14594189		-0.00077808		-144		+17
		-0.00342956		+6359		-60	
XI 28	+0.14251233		-0.00071449		-204		+18
		-0.00414405		+6155		-42	
XII 8	+0.13836828		-0.00065294		-246		+11
		-0.00479699		+5909		-31	
XII 18	+0.13357129		-0.00059385		-277		+11
		-0.00539084		+5632		-20	
XII 28	+0.12818045		-0.00053753		-297		

Date	^{II}y	^{I}y	f_y	Δ			
1938 VII 11	+0.08334109		-0.00105917				
		-0.00729190		+14097			
VII 21	+0.07604919		-0.00091820		-203		
		-0.00821010		+13894		-278	
VII 31	+0.06783909		-0.00077926		-481		+89
		-0.00898936		+13413		-189	
VIII 10	+0.05884973		-0.00064513		-670		+69
		-0.00963449		+12743		-120	
VIII 20	+0.04921524		-0.00051770		-790		+49
		-0.01015219		+11953		-71	
VIII 30	+0.03906305		-0.00039817		-861		+42
		-0.01055036		+11092		-29	
IX 9	+0.02851269		-0.00028725		-890		+23
		-0.01083761		+10202		-6	
IX 19	+0.01767508		-0.00018523		-896		+25
		-0.01102284		+9306		+19	
IX 29	+0.00665224		-0.00009217		-877		+12
		-0.01111501		+8429		+31	
X 9	-0.00446277		-0.00000788		-846		+13
		-0.01112289		+7583		+44	
X 19	-0.01558566		+0.00006795		-802		+6
		-0.01105494		+6781		+50	
X 29	-0.02664060		+0.00013576		-752		+7
		-0.01091918		+6029		+57	
XI 8	-0.03755978		+0.00019605		-695		+3
		-0.01072313		+5334		+60	
XI 18	-0.04828291		+0.00024939		-635		-2
		-0.01047374		+4699		+58	
XI 28	-0.05875665		+0.00029638		-577		+4
		-0.01017736		+4122		+62	
XII 8	-0.06893401		+0.00033760		-515		-5
		-0.00983976		+3607		+57	
XII 18	-0.07877377		+0.00037367		-458		+1
		-0.00946609		+3149		+58	
XII 28	-0.08823986		+0.00040516		-400		

Date	^{II}z	^{I}z	f_z	Δ			
1938 VII 11	+0.02883106		-0.00036768				
		-0.00796288		+11258			
VII 21	+0.02086818		-0.00025510		-883		
		-0.00821798		+10375		-82	
VII 31	+0.01265020		-0.00015135		-965		+59
		-0.00836933		+9410		-23	
VIII 10	+0.00428087		-0.00005725		-988		+33
		-0.00842658		+8422		+10	
VIII 20	-0.00414571		+0.00002697		-978		+30
		-0.00839961		+7444		+40	
VIII 30	-0.01254532		+0.00010141		-938		+10
		-0.00829820		+6506		+50	
IX 9	-0.02084352		+0.00016647		-888		+15
		-0.00813173		+5618		+65	
IX 19	-0.02897525		+0.00022265		-823		+1
		-0.00790908		+4795		+66	
IX 29	-0.03688433		+0.00027060		-757		+4
		-0.00763848		+4038		+70	
IX 9	-0.04452281		+0.00031098		-687		+1
		-0.00732750		+3351		+71	
X 19	-0.05185031		+0.00034449		-616		-2
		-0.00698301		+2735		+69	
X 29	-0.05883332		+0.00037184		-547		-3
		-0.00661117		+2188		+66	
XI 8	-0.06544449		+0.00039372		-481		-3
		-0.00621745		+1707		+63	
XI 18	-0.07166194		+0.00041079		-418		-3
		-0.00580666		+1289		+60	
XI 28	-0.07746860		+0.00042368		-358		-6
		-0.00538298		+931		+54	
XII 8	-0.08285158		+0.00043299		-304		-4
		-0.00494999		+627		+50	
XII 18	-0.08780157		+0.00043927		-254		

x	y	z	+0.09790668	+0.08325282	+0.02880045
r^2	Date	- h	0.017346216	VII 11	-123670,0
(x)	(y)	(z)	+4.06361	-2.69641	-1.25625
$(r)^2$		- (h)	25.36172		-2316,841
q	f	(h) f q	+57513,201	2.957436	+39,408
+0.10885338	+0.07597268	+0.02084696	+0.11853538	+0.06777417	+0.01263763
0.01805550	VII 21	-116454,8	0.018803688	VII 31	-109573,9
+4.10679	-2.63641	-1.23156	+4.14901	-2.57580	-1.20659
25.33312		-2320,767	25.30489		-2324,653
+90828,255	2.9332935	+61,831	+123053,4	2.910288	+83,251
+0.12692831	+0.05879600	+0.00427614	+0.13402951	+0.04917212	-0.00414342
0.019586051	VIII 10	-103074,5	0.020398975	VIII 20	-96974,8
+4.19026	-2.51459	-1.18134	+4.23054	-2.45280	-1.15582
25.27701		-2328,494	25.24962		-2332,290
+153798,34	2.888652	+103,448	+182734,0	2.868562	+122,255
+0.13985423	+0.03902988	-0.01253683	+0.14443238	+0.02848879	-0.02082961
0.021239709	VIII 30	-91274,3	0.022106196	IX 9	-85960,811
+4.26983	-2.39043	-1.13002	+4.30813	-2.32750	-1.10396
25.22255		-2336,043	25.19597		-2339,747
+209591,14	2.850148	+139,547	+234154,1	2.833499	+155,236
+0.14780496	+0.01765967	-0.02895665	+0.15002204	+0.00664461	-0.03686175
0.022996658	IX 19	-81016,677	0.023909552	IX 29	-76421,285
+4.34541	-2.26403	-1.07764	+4.38167	-2.20002	-1.05107
25.16973		-2343,405	25.14387		-2347,012
+256257,94	2.818671	+169,266	+275784,07	2.805693	+181,604
+0.15114007	-0.00446339	-0.04449686	+0.15122011	-0.01557995	-0.05182158
0.024843213	X 9	-72153,912	0.025795733	X 19	-68194,556
+4.41691	-2.13550	-1.02425	+4.45110	-2.07048	-0.99719
25.11854		-2350,569	25.09357		-2354,076
+292652,8	2.794571	+192,239	+306822,67	0.08545886	+201,177
+0.15032601	-0.02662926	-0.05880231	+0.14852310	-0.03754341	-0.06541166
0.026764738	X 29	-64524,853	0.027747304	XI 8	-61128,020
+4.48425	-2.00497	-0.96989	+4.51634	-1.93898	-0.94237
25.06909		-2357,526	25.04503		-2360,923
+318282,156	0.08841306	+208,436	+327047,87	0.09066207	+214,046
+0.14587706	-0.04826210	-0.07162769	+0.14245281	-0.05873192	-0.07743328
0.028739873	XI 18	-57988,813	0.029738154	XI 28	-55093,516
+4.54736	-1.87254	-0.91462	+4.57731	-1.80565	-0.88666
25.02142		-2364,264	24.99830		-2367,548
+333158,665	0.09222455	+218,043	+336673,3	0.09312106	+220,469
+0.13831388	-0.06890585	-0.08281548	+0.13352181	-0.07874262	-0.08776495
0.030737149	XII 8	-52429,549	0.031731160	XII 18	-49985,331
+4.60617	-1.73834	-0.85849	+4.63394	-1.67061	-0.83011
24.97563		-2370,777	24.95342		-2373,943
+337666,70	0.09337436	+221,370	+336226,62	0.09300722	+220,794
+0.12813567	-0.08820607	-0.09227537	+0.12221174	-0.09726473	-0.09634296
0.032713805	XII 28	-47750,169	0.033678103	I 7	-45714,088
+4.66060	-1.60249	-0.80154	+4.68616	-1.53399	-0.77277
24.93163		-2377,050	24.91039		-2380,095
+332452,29	0.09204399	+218,793	+326448,88	0.09050870	+215,419

the perturbations in the elements. Subsequently, B. Stroemgren published a similar method in the Publications of the Kobenhavn Observatory, No. 65, which is developed in rectangular coordinates and is better adapted to machine computation. We shall give this latter method in essentially the form in which it was presented by Stroemgren.

Consider the elements of the elliptic orbit which will be derived from a position vector and a velocity vector by means of the formulas on page 47 or their equivalent. As t changes, these vectors also change, but the elements which we derive from them remain fixed as long as the Sun is the only attracting body. If there is as additional force, m**U**, acting, it will cause changes in these two vectors of a different kind, and these changes will produce changes in the elements that are derived from them. It is these changes, due to the attractive force of a disturbing planet, which we wish to determine.

The effect of the acceleration **U** on **r** and **v** will be $d\mathbf{r} = \frac{1}{2}\mathbf{U}_0 dt^2$ and $d\mathbf{v} = \mathbf{U}_0 dt$ plus terms of higher order. Neglecting terms of higher order than the first, we have $d\mathbf{r} = 0$, $d\mathbf{v} = \mathbf{U}dt$, which is acting at every instant and tending to change the osculating elements. Consider a coordinate system with the origin at the Sun, the x-axis directed toward the perihelion of the osculating orbit of the minor planet at t_0, the y-axis directed toward $v = 90°$ in the orbit plane, and the z-axis directed toward the normal of the orbit plane. Then $\mathbf{r}\times\mathbf{v} = k\sqrt{p}\,\mathbf{N}$, where **N** is the unit vector that is normal to the orbit plane. By differentiation and substitution

$$d(k\sqrt{p}\,\mathbf{N}) = \mathbf{r}\times\mathbf{U}\,dt \qquad ((7,14))$$

To terms of the first order, the component of this equation along the normal will give an equation for the variation of the magnitude of $\mathbf{r}\times\mathbf{v}$ due to **U**, and the components along the two axes in the orbit plane will give the variation of the direction of the normal. Thus

$$dp = \frac{2\sqrt{p}}{k}(xU_y - yU_x)\,dt, \quad dN_x = \frac{yU_z}{k\sqrt{p}}dt = d\Theta_y, \quad dN_y = -\frac{xU_z}{k\sqrt{p}}dt = -d\Theta_x, \qquad ((7,15))$$

where Θ_x and Θ_y represent the rotations about the x and y axes, respectively.

From the equation $\mathbf{v}\cdot\mathbf{v} = k^2(2/r - 1/a)$, we obtain by differentiation and substitution

$$2\mathbf{v}\cdot\mathbf{U}dt = \frac{k^2}{a^2}da. \qquad ((7,16))$$

Substituting from ((4,20)), we obtain

$$da = \frac{2a^2}{rk\sqrt{p}}\Big\{(x + er)U_y - yU_x\Big\}dt \qquad ((7,17))$$

The angle between **r** and the fixed x-axis is not affected by **U**, since $d\mathbf{r} = 0$, therefore $d(\pi + v) = 0$ or $d\pi = -dv$. Also $e\cos v = \frac{p}{r} - 1$ and $e\sin v = \frac{\sqrt{p}\,(\mathbf{r}\cdot\mathbf{v})}{rk}$.

Again by differentiation and substitution

$$\cos v\,de - e\sin v\,dv = \frac{dp}{r}$$

$$\sin v\,de + e\cos v\,dv = \frac{e\sin v}{2p}dp + \frac{\sqrt{p}\,(\mathbf{r}\cdot\mathbf{U})\,dt}{rk}$$

Then

$$de = \frac{x}{r^2}dp + \frac{e\sin^2 v}{2p}dp + \frac{\sqrt{p}\,y\,(\mathbf{r}\cdot\mathbf{U})}{r^2k}dt$$

$$= \frac{\sqrt{p}}{r^2k}\Big\{y(xU_x + yU_y) + \Big(2x + \frac{ey^2}{p}\Big)(xU_y - yU_x)\Big\}dt \qquad ((7,18))$$

$$= \frac{\sqrt{p}}{k}\Big[U_y + (xU_y - yU_x)\Big(\frac{x + ea}{ra}\Big)\Big]dt$$

Also

$$-e\,dv = \frac{y}{r^2}dp - \frac{exy}{2pr^2}dp - \frac{\sqrt{p}\,x}{r^2k}(xU_x + yU_y)\,dt$$

$$
\begin{aligned}
-e\,dv &= \frac{2\sqrt{p}}{k}\left(\frac{y}{r^2} - \frac{e\,x\,y}{2\,p\,r^2}\right)(x\,U_y - y\,U_x)\,dt - \frac{\sqrt{p}\,x}{r^2 k}(x\,U_x + y\,U_y)\,dt \\
&= \frac{\sqrt{p}}{r^2 k}\left[U_y\,x\,y\left(1 - \frac{e\,x}{p}\right) - U_x\left(r^2 + y^2\left(1 - \frac{e\,x}{p}\right)\right)\right]dt \qquad ((7{,}19)) \\
&= \frac{\sqrt{p}}{k}\left[\frac{y}{r\,p}(x\,U_y - y\,U_x) - U_x\right]dt
\end{aligned}
$$

The equation $M = M_0 + n\,(t - t_0) = E - e\sin E$ is affected in several ways by **U**. We have

$$dM = (1 - e\cos E)\,dE - \sin E\;de$$

and from ((4,19)) it can be shown that $dE = \frac{r}{b}dv - \frac{y}{b\,(1 - e^2)}de.$

Therefore

$$dM_0 = \frac{r^2}{a\,b}dv - \frac{r\,y}{a\,b\,(1 - e^2)}de - \frac{y}{b}de. \qquad ((7{,}20))$$

In order to deal with a smaller quantity (i.e. the difference of two quantities of nearly equal magnitude) let

$$dL_1 = dM_0 + d\pi;\;\text{ then }\; dL_1 = \left(1 - \frac{r^2}{a\,b}\right)d\pi - \frac{y}{b}\left(1 + \frac{r}{p}\right)de. \qquad ((7{,}21))$$

But M is also affected by the changes produced by **U** on n, the mean motion. If we write

$$dM = dL_1 - d\pi + (n_0 + dn)\,dt,$$

then

$$M = M_0 + \int\frac{dL_1}{dt}dt - \int\frac{d\pi}{dt}dt + n_0\,(t - t_0) + \iint\frac{dn}{dt}dt^2 \qquad ((7{,}22))$$

The latter double integral will be represented by ΔL_n. Also

$$dn = -\frac{3\,n}{2\,a}da = \frac{-\,3}{r\sqrt{a\,p}}\left[(x\,U_y - y\,U_x) + e\,r\,U_y\right]dt. \qquad ((7{,}23))$$

Finally, $\frac{da}{a} = -\frac{2}{3}\frac{dn}{n}$, and this will be obtained from the single integral of $\frac{dn}{dt}$.

The coordinates of the disturbing planet will be expressed in terms of our present coordinate system by $\xi = \mathbf{P}\cdot\mathbf{r}_1$, $\eta = \mathbf{Q}\cdot\mathbf{r}_1$, $\zeta = \mathbf{R}\cdot\mathbf{r}_1$, where $\mathbf{r}_1$ and the vectorial constants may be referred to the equatorial coordinate system of, say, 1950.0 In actual computation, the components of **P**, **Q**, and **R** should be written on a slip of paper which fits directly under the columns of coordinates of Jupiter in the volumes of Planetary Coordinates. Then

$$x = a(\cos E - e), \quad y = b\sin E, \quad r^2 = x^2 + y^2, \quad \rho^2 = r^2 + r_1^2 - 2\,(x\xi + y\eta). \qquad ((7{,}24))$$

If the numerical integrations are to be computed with an interval of w mean solar days (usually w = 80), and if m is the mass of the disturbing planet, let $k_1 = m\,w\,k/\sqrt{p}$, $k_2 = 3\,w\,k/\sqrt{a}$, $k_5 = e\,p$, $k_6 = (1 - e^2)$, $k_7 = p$, $k_{10} = 1/e\,a\,b$, $k_{11} = 1/b$, $k_{12} = 1/b\,p$. Then the components of **U**, multiplied by convenient factors, are

$$
\begin{aligned}
L &= k_1(1/\rho^3 - 1/r_1^3)\xi - (k_1/\rho^3)\,x \\
M &= k_1(1/\rho^3 - 1/r_1^3)\eta - (k_1/\rho^3)\,y \qquad ((7{,}25)) \\
N &= k_1(1/\rho^3 - 1/r_1^3)\zeta
\end{aligned}
$$

Also let $T = (M\,x - L\,y)/r$. The reader will not confuse this use of the notation, L, M, N, and T, since it is confined to the following collection of formulas. The necessary equations for the computation of the function columns of the various integrals are:

$$
\begin{aligned}
&w\frac{d\Theta_x}{dt} = x\,N, \qquad w\frac{d\Theta_y}{dt} = y\,N, \qquad w^2\frac{dn}{dt} = -\,k_2(T + e\,M), \\
&w\frac{de}{dt} = (k_5 + k_6 x)\,T + k_7 M, \qquad w\,e\frac{d\pi}{dt} = y\,T - k_7 L, \qquad ((7{,}26)) \\
&w\frac{dL_1}{dt} = -\,(k_{11} + k_{12} r)\,y\,w\frac{de}{dt} + (1/e - k_{10} r^2)\,w\,e\frac{d\pi}{dt}.
\end{aligned}
$$

The computations for any one date may be conveniently arranged as follows:

Date	JD	M	r_1	ρ^2
ξ	η	ζ	$1/r_1^3$	$k_1(1/\rho^3 - 1/r_1^3)$
x	y	r^2	r	$- k_1(1/\rho^3)$
L	M	N	T	$(1/e - k_{10}r^2)$

The quantities, x and y, may be obtained either by the direct computation of Kepler's equation or, with sufficient accuracy, from either the Appendix to the Union Observatory Circular No. 71 or the Veroffentlichungen des Rechen-Instituts, No. 46. Two simple checks may be applied, in addition to the smoothness of the differences of the functions in the integration tables. First, the values of r_1 in the first row and fourth column are copied down directly from Planetary Coordinates. Then $(\xi^2 + \eta^2 + \zeta^2)/r_1 = r_1$ is a check on the computation of Jupiter's transformed coordinates. Second, compute the value of r in the third row and fourth column from a specially constructed table, so that at the time r^2 is computed it may be divided by this independent value of r to give a quotient of r, which is a check on x and y. The specially constructed table gives r as a function of M at intervals of 10°. It is constructed by means of the numerical integration of the differential equation

$$\frac{d^2r}{dM^2} = \frac{(p - r)}{r^3}a^3. \qquad (7,27)$$

Using Cowell's original method, we have

$$f(r) = \frac{(p - r)}{r^3}a^3(\text{arc } 10°) = \frac{2.56713 - r}{r^3}0.553133 \qquad (7,28)$$

in the present case. The complete table is shown below. One begins its construction at M = 0 with r = a(1 - e), and takes advantage of the fact that the table is symmetrical about this point. The completed table is checked by the accuracy with which it "closes" at M = 180°, i.e. r = a(1 + e).

As an illustration, we give some of the computations for (1531) = 1938 SH. The elements were taken from the 1944 Kleine Planeten. The vectorial constants were computed according to the precepts at the bottom of page 50. These are followed by the auxiliaries which are needed. The integration tables have been adjusted so that the perturbations osculate at 1938 Oct. 29.0 UT., and they are computed in units of the 7th decimal of a radian.

Epoch 1938 Oct. 29.0 UT = JD 2429200.5

M	322°.909	n	0°.231292
i	12.394	a	2.6284
Ω	279.440	ϕ	8.782
ω	141.230	e	0.152675
	(1950)	b	2.5976

+0.2146330	+0.9766948	
+0.1601923	+0.6261957	+0.1640147
-0.9864579	-0.7796659	-0.9634683
+0.8694193	+0.3978812	+0.4928192
+0.1344023	+0.9174370	-0.1673420

P	Q	R
+0.475443	-0.853889	-0.211726
+0.744161	+0.518713	-0.420905
+0.469231	+0.042558	+0.882049

k_1	0.0008200,7	k_7	2.56713
k_2	2.54653	k_{10}	0.95938
k_5	0.39194	k_{11}	0.38497
k_6	0.97669	k_{12}	0.14996

M	r	Δ^{I}	Δ^{II}
0°	2.2271	+ 85	+170
10	2.2356	+248	+163
20	2.2604	+394	+146
30	2.2998	+515	+121
40	2.3513	+607	+92
50	2.4120	+668	+61
60	2.4788	+701	+33
70	2.5489	+707	+ 6
80	2.6196	+691	-16
90	2.6887	+657	-34
100	2.7544	+608	-49
110	2.8152	+546	-62
120	2.8698	+476	-70
130	2.9174	+398	-78
140	2.9572	+314	-84
150	2.9886	+228	-86
160	3.0114	+137	-91
170	3.0251	+ 46	-91
180	3.0297		-92

'38 VII 1	9080.5	-64.847	5.0390	7.7806
-0.7406	-4.9162	-0.8207	78160,	+313,76
+0.3574	-2.4870	6.3129	2.5126	-377,86
-367,42	-602,77	-257,50	-449,42	+0.4939

'38 IX 19	9160.5	-46.343	5.0170	10.6136
-0.1245	-4.9307	-0.9176	79192,	+172,23
+1.1672	-2.0843	5.7067	2.3889	-237,17
-298,27	-354,88	-158,04	-433,63	+1.0755

'38 XII 8	9240.5	-27.839	4.9976	14.8150
+0.4935	-4.8714	-1.0008	80117,	+78,11
+1.8143	-1.3974	5.2444	2.2902	-143,81
-222,37	-179,54	-78,17	-277,91	+1.5190

'39 II 26	9320.5	-9.336	4.9811	20.2943
+1.1040	-4.7382	-1.0688	80914,	+23,34
+2.1788	-0.4959	4.9931	2.2345	-89,70
-169,67	-66,11	-24,95	-102,12	+1.7600

'39 V 17	9400.5	+9.167	4.9679	26.6840
+1.6976	-4.5324	-1.1204	81562,	-7,39
+2.1805	+0.4871	4.9918	2.2343	-59,49
-142,26	+4,52	+8,28	+35,43	+1.7613

'39 VIII 5	9480.5	+27.671	4.9581	33.4179
+2.2650	-4.2566	-1.1547	82047,	-24,83
+1.8191	+1.3901	5.2415	2.2895	-42,45
-133,46	+46,68	+28,67	+118,12	+1.5217

'39 X 24	9560.5	+46.174	4.9519	39.9349
+2.7972	-3.9147	-1.1711	82354,	-35,04
+1.1740	+2.0793	5.7018	2.3879	-32,50
-136,17	+69,59	+41,04	+152,79	+1.0802

'40 I 12	9640.5	+64.677	4.9495	45.8563
+3.2859	-3.5118	-1.1693	82476,	-41,23
+0.3652	+2.4847	6.3071	2.5113	-26,41
-145,12	+79,17	+48,21	+155,10	+0.4995

JD	Θ_x					Θ_y					ΔL_n $w\Delta n$					
9080		-92					+640				+1256		+1379			
	+182		-92			-320		-311				-1222		-137		
9160		-184		+134			+329		+91		+34		+1242		-327	
	-2		+42		-88	+9		-220		+32		+20		-464		+299
9240		-142		+46			+109		+123		+54		+778		-28	
	-144		+88		-62	+118		-97		-34		+798		-492		+142
9320		-54		-16			+12		+89		+852		+286		+114	
	-198		+72		-22	+130		-8		-45		+1084		-378		+37
9400		+18		-38			+4		+44		+1936		-92		+151	
	-180		+34		0	+134		+36		-35		+992		-227		-21
9480		+52		-38			+40		+9		+2928		-319		+130	
	-128		-4		+12	+174		+45		-19		+673		-97		-43
9560		+48		-26			+85		-10		+3601		-416		+87	
	-80		-30		+13	+259		+35		-11		+257		-10		-37
9640		+18		-13			+120		-21		+3858		-426		+50	

JD	Δe					$e\Delta\pi$					ΔL_1				
9080		-1880					+2061					-2545			
	+1553		+305			-1639		-391			+602		+1901		
9160		-1575		+208			+1670		-320			-644		-881	
	-22		+513		-86	+31		-711		+558	-42		+1020		+188
9240		-1062		+122			+959		+238			+376		-693	
	-1084		+635		-229	+990		-473		+131	+334		+327		+301
9320		-427		-107			+486		+369			+703		-392	
	-1511		+528		-146	+1476		-104		-140	+1037		-65		+210
9400		+101		-253			+382		+229			+638		-182	
	-1410		+275		+16	+1858		+125		-194	+1675		-247		+119
9480		+376		-237			+507		+35			+391		-63	
	-1034		+38		+104	+2365		+160		-104	+2066		-310		+67
9560		+414		-133			+667		-69			+81		+4	
	-620		-95		+100	+3032		+91		-16	+2147		-306		+29
9640		+319		-33			+758		-85			-225		+33	

To compute a position for any time, t, interpolate the values of all the integrals for this time and then $\frac{da}{a} = \frac{(w\,\Delta n)}{1.5\,w\,n}$, where the numerator and denominator must be expressed in the same units; $M = M_0 + n_0(t - t_0) + \Delta L_1 - \Delta\pi + \Delta L_n$, and solve Kepler's equation, using $e = e_0 + \Delta e$. Let

$$\Psi = \Theta_x \mathbf{P} + \Theta_y \mathbf{Q} + \Delta\pi\,\mathbf{R} \tag{7,29}$$

and this must be applied to **P** and **Q** in the manner described on page 85. Then the semi-major and semi-minor axes must be adjusted to the new values of $a_0(1 + da/a)$ and $e_0 + \Delta e$, and finally

$$\mathbf{r} = a\,\mathbf{P}\,(\cos E - e) + b\,\mathbf{Q}\sin E \tag{7,30}$$

When it is desired to compute the usual opposition ephemeris, the date of opposition may be determined by constructing a table in which the argument is the mean anomaly, M, and the function is Arctan (y/x). If one finds a date such that the mean anomaly of the minor planet on that date gives a respondent from the table which coincides with the "Right Ascension of the Mean Sun plus 12 hours" for that same date, as determined from the annual ephemeris of the Sun, then that date is approximately the date of opposition. Such a table is shown below for (1531), and it indicates that in 1947 the opposition date was approximately December 8th. The perturbations for that date are also shown, and such parts of the computation as have not been previously illustrated. The student may complete the computations as an exercise.

It is very valuable to the observer in identifying an object, if the computer provides the Variation. This is the expected change in the position of the object on the sky, if the object were to arrive earlier (or later) at some given position in its orbit. One method of computing the Variation is to combine the solar coordinates for some ephemeris date, t_1, with the planet's rectangular coordinates as computed for the next ephemeris date, and designate the resulting position on the sky by the subscript v. Then

$$\text{Variation} = \frac{(\delta_v - \delta_1)'}{(\alpha_v - \alpha_1)^m} \tag{7,31}$$

and it is best to give the numerator and denominator separately. Then any estimate of the error of the mean anomaly can be immediately converted into an estimate of the uncertainty of the position of the object on the sky.

Another important datum to aid in the identification of an observed object is the magnitude. Due to the way in which astronomical magnitudes are defined, and assuming the inverse square law for the diminution of brightness, the magnitude is given by the formula

$$\text{Mag.} = g_0 + 5(\log\rho + \log r). \tag{7,32}$$

This formula is not satisfactory for comets, because their intrinsic brightness appears to increase as they approach the Sun; they do not shine simply be reflected sunlight. The formula may be modified to read

$$\text{Mag.} = g_0 + 5(\log\rho + \tfrac{1}{2}n\log r) \tag{7,33}$$

where n often lies within the range from 4 to 6. The formula (7,32) is also not satisfactory for minor planets if the illuminated portion of the reflecting surface is not turned directly toward the observer. The correction for this "phase angle" requires that the formula be written

$$\text{Mag.} = g_0 + 5(\log\rho + \log r) + C\beta \tag{7,34}$$

where $\beta = \arccos[x(x+X) + y(y+Y) + z(z+Z)]/r\rho$. All the coefficients, g_0, n, and C, must be determined empirically from previous observations, and separately for each object.

1947 Dec. 8

Θ_x	-0.000228
Θ_y	+0.000422
da/a	+0.000041
Δe	+0.001585
$e\Delta\pi$	+0.000480
$\Delta\pi$	+0.00318
ΔL_n	+0.000355
ΔL_1	-0.001472
ΔM	$-0^\circ\!.2462$

ψx

$$\begin{Bmatrix} 0.0 & +0.002716 & +0.001289 \\ -0.002716 & 0.0 & -0.001142 \\ -0.001289 & +0.001142 & 0.0 \end{Bmatrix}$$

ΔP		ΔQ	
-0.002626	-2	-0.001464	+4
+0.001827	-4	-0.002271	-3
-0.000237	-3	-0.001693	0

ψ	P	Q
-0.001142	+0.472815	-0.855349
-0.001289	+0.745984	+0.516439
+0.002716	+0.468991	+0.040865

a 2.62850 b 2.59704
e 0.15426 e $8^\circ\!.8384$

M	$\alpha_\odot + 12^h$	var./deg.
0°	$3^h\ 49^m\!.7$	$6^m\!.08$
30	6 47.9	5.38
60	9 06.8	4.05
90	10 55.8	3.36
120	12 33.5	3.20
150	14 10.1	3.26
180	15 49.7	3.36
210	17 31.1	3.38
240	19 12.0	3.34
270	20 53.2	3.44
300	22 42.6	3.97
330	0 56.5	5.17
360	3 49.7	6.08

1947 UT	Nov. 23.0 UT	Dec. 1.0 UT
M	$+8^\circ\!.7019$	+10.5522
E	+10.27905	+12.45905
Cos E	+0.983950	+0.976450
Cos E - e	+0.829690	+0.822190
Sin E	+0.178442	+0.215742
r cos v	+2.18084	+2.16113
r sin v	+0.46342	+0.56029

Ephemeris

Nov.	23	$5^h\ 30^m\!.6$		+32° 44′		
			7.6		44	
Dec.	1	5 23.0		+32 00		
			8.6		56	
	9	5 14.4		+31 04		
	12		8.7		66	
	17	5 05.7		+29 58		
			7.8		72	
	25	4 57.9		+28 46		
			6.4		73	
Jan.	2	4 51.5		+27 33		

13.0 = mag. $g_o = 11.2$

0.350 = log r

$-41'$
$+20^m\!.6$ = var.

0.100 = log ρ

CHAPTER 8

HANSEN'S METHOD OF GENERAL PERTURBATIONS *

Ἐνθάδε ὑμῖν ἐστι τὸ Ἄλφα καὶ τὸ Ὦ μέγα.

The minor planets present a formidable problem in Astronomy. Their total number has been estimated to be very large, about 50,000. Over a period of many years, their paths about the Sun are complicated by the attractions of the planets. These increase the labor of prediction and identification. More complex data are also necessary to give a truly descriptive characterization of their motions. One mode of procedure is to generalize the expressions for simple, two-body, elliptic motion about the Sun so as to include the disturbing effects of the planets by utilizing infinite, trigonometric series; the resulting expressions are known as General Perturbations.

Perhaps the one method which is most generally and expeditiously applicable to all cases, especially those of large eccentricity and inclination, is that of Prof. P. A. Hansen. This method is developed in great detail in the "Auseinandersetzung" ** and includes an example. But the material is not elementary, and to the beginner it must appear very imposing. Without a mastery of the whole subject, it is difficult to discern the real essentials from the wealth of detailed theoretical development. One cannot "see the forest because of the trees". In the hope that a simple outline of the actual processes involved, with all the necessary formulas, with adaptations to modern methods of machine computing, and a detailed description of each step of the computation, all fully illustrated, will make this field of research more inviting to the uninitiated, the author presents the following material.

No attempt will be made to develop even the simplest foundations of the theory. All such material will be taken directly from other sources whenever possible, especially the Auseinandersetzung. Hansen's method depends essentially upon the use of a fixed elliptic orbit as a basis, and then defines the necessary components of a displacement from the position which the minor planet would have in this ellipse to its disturbed position in space. The longitude, l, and the latitude, b, referred to any fixed fundamental plane, are given by:

$$\begin{aligned}
\cos b \sin(l - \theta - \Gamma) &= \cos i_0 \sin(v - \theta) - s(\tan i_0 + q/\kappa \cos i_0) \\
\cos b \cos(l - \theta - \Gamma) &= \cos(v - \theta) + s p/\kappa \\
\sin b &= \sin i_0 \sin(v - \theta) + s
\end{aligned} \qquad ((8,1))$$

(Ausein. I: 79, (21)).

In these equations, i_0 and θ are the inclination and node, respectively, of the fixed elliptic orbit upon the fundamental plane. Γ, s p, and s q are second order perturbations. If these are neglected and the orbit plane is adopted as the fundamental plane, then these equations reduce to:

$$\begin{aligned}
\cos b \cos l &= \cos v \\
\cos b \sin l &= \sin v \\
\sin b &= s
\end{aligned} \qquad ((8,2))$$

* The contents of this chapter were prepared in manuscript about 1939, but no previous occasion for their publication has ever presented itself. The example has been chosen from about a dozen cases which were worked out at that time, because it is fairly representative, neither too simple nor too difficult. Since this chapter was prepared independently of the previous chapters, there may be some repetition in the text.

** Hansen: Auseinandersetzung einer zweckmassigen Methode zur Berechnung des Absoluten Storungen der kleinen Planeten. Abhand. I, II, III. Abhand. der K. S. Gesell. der Wissen. V-VII

The quantity s is the component of the perturbation normal to the orbit plane. The component along the radius vector is ν, defined by $r = \bar{r}(1 + \nu)$. The angle v is not the true orbital longitude in the usual sense, but includes the effect of the component of the perturbation in the orbit plane and at right angles to the radius vector, namely ndz. Thus

$$\begin{aligned} v &= \bar{f} + \pi_0 \\ \bar{r}\cos\bar{f} &= a_0(\cos\bar{E} - e) \\ \bar{r}\sin\bar{f} &= a_0\cos\phi\,\sin\bar{E} \\ \bar{E} - e_0\sin\bar{E} &= M_0 + n_0(t - t_0) + ndz. \end{aligned} \tag{(8,3)}$$

(Ausein. I: 91)

Since it is very important to have a clear understanding of this fundamental basis of Hansen's method, the description will be reiterated. Suppose the fixed, elliptic orbit is known (the elements are denoted by the subscript $_0$); three components of the perturbations, ndz, ν, and u ($= \bar{r}\, s/a_0$) are known; and the second order perturbations are neglected. This latter condition shall be understood as applying to the perturbations whenever they are referred to hereafter. The position in the fixed elliptic orbit at any time t would be found by solving Kepler's equation:

$$M_0 + n_0(t - t_0) = E - e_0\sin E$$

Then

$$\mathbf{r} = \mathbf{A}(\cos E - e_0) + \mathbf{B}\sin E \tag{(8,4)}$$

But the disturbed position in space is found as follows: solve Kepler's equation with a disturbed mean anomaly, $M_0 + n_0(t - t_0) + ndz = \bar{E} - e_0\sin\bar{E}$. This is sometimes referred to as a perturbation in the time, since it is equivalent to solving Kepler's equation for a time (t + dz) instead of t. The eccentric anomaly $\bar{E}$ now corresponds to the projection of the disturbed position upon the fixed orbit plane. The length of the disturbed radius vector when projected onto the fixed orbit plane will be $a_0(1 - e_0\cos\bar{E})(1 + \nu) = \bar{r}(1 + \nu)$. The displacement normal to the fixed orbit plane is $r\sin b = u\,a_0(1 + \nu)$. Combining these results gives:

$$\mathbf{r} = [\mathbf{A}(\cos\bar{E} - e_0) + \mathbf{B}\sin\bar{E} + \mathbf{C}u](1 + \nu) \tag{(8,5)}$$

Elementary vector notions have been introduced because they afford a better geometrical interpretation of the quantities involved, they simplify the notation, and even the computations to some extent. The notations used here are defined in Planetary Coordinates (London 1933,1939) and by Smiley in Astronomical Journal, v 40, p 31. The latter also contains a valuable bibliography. Let P_x, P_y, P_z be the direction cosines of the half line directed from the Sun to the perihelion of the minor planet orbit; then **P** is a unit vector extending from the Sun to the point whose rectangular coordinates are P_x, P_y, P_z.

Similarly, if **R** is the unit vector normal to the orbit plane, then R_x, R_y, R_z are the direction cosines of the normal. Conventionally, the normal vector is so directed that the revolution of the planet will appear to be counterclockwise if viewed from any point along the positive direction of **R**. Finally, **Q** is a unit vector mutually perpendicular to **P** and **R** and directed toward the position in the orbit plane 90° in advance of the perihelion.

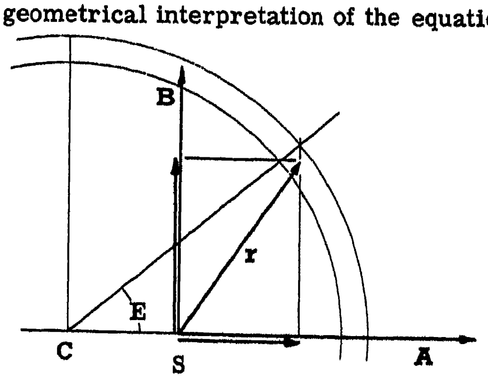
Now define $\mathbf{A} = a\mathbf{P}$, $\mathbf{B} = a\cos\phi\,\mathbf{Q}$, $\mathbf{C} = a\mathbf{R}$. These new vectors will have the same directions respectively as the old ones, but different lengths. The geometrical interpretation of the equation

$$\mathbf{r} = \mathbf{A}(\cos E - e_0) + \mathbf{B}\sin E$$

is shown in the diagram. This vector equation is equivalent to the three scalar equations

$$\begin{aligned} x &= A_x(\cos E - e_0) + B_x\sin E \\ y &= A_y(\cos E - e_0) + B_y\sin E \\ z &= A_z(\cos E - e_0) + B_z\sin E \end{aligned}$$

which are its components upon the coordinate axes.



The interpretation of the equation

$$\mathbf{r} = [\mathbf{A}(\cos\bar{E} - e_0) + \mathbf{B}\sin\bar{E} + \mathbf{C}u](1 + \nu)$$

is similar except that the Cu component places the terminal point of $\mathbf{r}$ above the plane of the paper (below if u is negative), and the vector must be "stretched" by a factor $(1 + \nu)$, ("shrunk if ν is negative).

The quantities, ndz, ν, and u, are obtained as double Fourier series whose angular arguments depend upon the positions of the disturbed and disturbing planets in their respective orbits, and therefore depend implicitly upon the time. These series are derived by the following sequence of formulas:

$$\begin{aligned}
3\,a\Omega &= 3\,m'a\Delta^{-1} + (-3H) \\
a\,r\frac{\partial\Omega}{\partial r} &= m'a\Delta^{-3}\left(\frac{r'^2 - r^2}{2}\right) - \frac{1}{6}(3\,a\Omega) + \tfrac{1}{2}(-3H) \\
a^2\frac{\partial\Omega}{\partial Z} &= m'a(\Delta^{-3} - r'^{-3})[(\mathbf{C}\cdot\mathbf{A}')(\cos E' - e') + (\mathbf{C}\cdot\mathbf{B}')\sin E'] \\
&= m'a(\Delta^{-3} - r'^{-3})Z' \\
T &= \frac{1}{3}M\frac{\partial(3\,a\Omega)}{\partial E} + N\,a\,r\frac{\partial\Omega}{\partial r} \\
W &= \int T\,dE \\
R &= \int Q\,a^2\frac{\partial\Omega}{\partial Z}dE \\
ndz &= \int \overline{W}(1 - e\cos E)dE \\
\nu &= -\frac{1}{6}X_0 - \frac{e}{6}X_1 - \frac{1}{2}\overline{W} \\
u &= \overline{R}
\end{aligned} \tag{(8,6)}$$

where X_0 and X_1 are portions of the W series. See: Ausein. I: p. 106, (48); p. 119, line 4; p. 118, line 13; p. 123, line 16; p. 124, (59); p. 125, (61); p. 98, (40); p. 116, line 7. The derivation of ν and the significance of the $\overline{}$ will be explained later. Δ is the distance between the disturbed and the disturbing planets.

It will become evident that certain features which exist in Hansen's method are very fortunate, for the results are built upon the two relatively large Fourier series, Δ^{-1} and Δ^{-3}, which are obtained by harmonic analysis, and all the multiplications and transformations are performed upon these two series by means of relatively small series which are known at the start. It may also be noted, in passing, that since the actual distances, radius vectors, etc. cannot be determined before the perturbations are known, a process of successive approximations must be followed. If it is assumed that both the disturbed and disturbing planets move in their respective, fixed, elliptic orbits, then the first order perturbations can be derived. Taking into account the displacements of both planets as given by their first order perturbations yields more approximate values of the distances, radius vectors, etc. and gives rise to the second order perturbations. Taking into account the second order produces the third order, etc. Theoretically, this process continues *ad infinitum* and should include all the major planets; practically, the first order perturbations by Jupiter are adequate for the identification of most minor planets, second order perturbations by Jupiter and first order by Saturn will usually give a very high degree of accuracy, and some third order perturbations would be needed only for the most exacting problems, provided the orbit is not a case of near-commensurability.

Since there will, in most cases, be no *a priori* knowledge concerning the general motion of a minor planet, the fixed elliptic orbit to be used must be the planet's osculating orbit at some suitable epoch. In order to be sufficiently accurate, this osculating orbit should be based upon observations in at least four or five different oppositions, and should include the effects of the special perturbations over the interval covered by the observations.

The example which has been chosen for purposes of illustration is the minor planet (1286) Banachiewizca. The elements which are adopted have been deduced from observations in the five oppositions 1928, 1933, 1935, 1936, and 1937. The special perturbations due to Jupiter and Saturn were computed by Cowell's method, using an augmented mass of the Sun. The elements which are

adopted for Jupiter were taken from the Astronomical Papers of the American Ephemeris, v. 7, p. 23. For similar purposes in the future, it will be better to adopt elements in accordance with the precepts given by Clemence in the Astronomical Journal, v. 52, p. 89. To simplify the typographical composition, all the computations are given at the end of the chapter and they are labelled by the sheet numbers of the original computing sheets, without regard to the present page numbers.

The Fourier series which will be encountered in first order perturbations are of the form:

$$\sum_{i=-\infty}^{\infty}\sum_{j=0}^{\infty}(c,i,j)\cos(iE - jE') + (s,i,j)\sin(iE - jE') \qquad ((8,7))$$

It is also possible to use the mean anomaly, $(ig - jg')$, as the argument instead of the eccentric anomaly, and this has certain advantages, but the principal disadvantage is loss of rapid convergence, except for very small eccentricities. For actual computation, the series will be arranged as follows; the cos and sin factors are not written, but they must be understood to be associated with their corresponding coefficients:

	cos	sin	cos	sin	cos	sin	cos	sin	
i \ j	0		1		2		3		
....			.....	.....	.....	.....	.....	.....	.....
-2			(c,-2,1)	(s,-2,1)	(c,-2,2)	(s,-2,2)	(c,-2,3)	(s,-2,3)	.....
-1			(c,-1,1)	(s,-1,1)	(c,-1,2)	(s,-1,2)	(c,-1,3)	(s,-1,3)	.....
0	2(c,0,0)/2	0	(c,0,1)	(s,0,1)	(c,0,2)	(s,0,2)	(c,0,3)	(s,0,3)	.....
1	(c,1,0)	(s,1,0)	(c,1,1)	(s,1,1)	(c,1,2)	(s,1,2)	(c,1,3)	(s,1,3)	.....
2	(c,2,0)	(s,2,0)	(c,2,1)	(s,2,1)	(c,2,2)	(s,2,2)	(c,2,3)	(s,2,3)	.....
....	.....	.....	.....	.....	.....	.....	.....	.....	.....
....	.....	.....	.....	.....	.....	.....	.....	.....	.....

Now these coefficients, even though they have been written as continuing indefinitely in three directions, must, for both theoretical and practical reasons, converge, i.e. all the coefficients beyond certain limits of i and j must be less than some preassigned number or adopted degree of accuracy. The magnitudes of these coefficients will be controlled largely by the following conditions. Other things being equal, a greater eccentricity will produce less rapid convergence along values of $\pm i$. The directions of the perihelia also enter here, because the real effect is an eccentricity relative to the orbit of the disturbing planet. If the perihelia are oppositely directed, the relative eccentricity is increased, and *vice versa*. Again, other things being equal, a larger value of a/a' (or actually $a(1 + e)/a'$) will produce less rapid convergence along values of j. It is one of the advantages of Hansen's method that, in spite of the presence of these difficulties in any individual case, the work may, nevertheless, be carried to any desired degree of accuracy by increasing the number of points on the circle of partition in the harmonic analysis, or by continuing the computations for larger values of j, respectively, as the case may be. The effects of a greater inclination are not so simply stated. The magnitude of all the coefficients of u are increased, since they are proportional to $\sin J$. But a greater inclination may actually increase the rapidity of convergence along values of j (since it may increase the minimum aphelion approach of the minor planet to the disturbing planet), while it usually decreases the rapidity of convergence along values of $\pm i$. Finally, the value of n'/n, the ratio of the mean motions, is an important controlling factor, since near-commensurabilities will produce small divisors during the integration process and thus increase the magnitude and reduce the accuracy of certain coefficients. In nearly all cases, this difficulty can be counteracted by deriving these coefficients to a larger number of decimal places before the integration is performed, so that no accuracy is wanting in the final result. However, these larger coefficients may then produce a relatively larger effect upon the higher order perturbations, perhaps to such an extent that they can no longer be neglected.

To derive the series for Δ^{-1} and Δ^{-3}, write

for the disturbed planet $\quad \mathbf{r} = \mathbf{A}(\cos E - e_0) + \mathbf{B}\sin E$

for the disturbing planet $\quad \mathbf{r}' = \mathbf{A}'(\cos E' - e') + \mathbf{B}'\sin E'$

$$\Delta^2 = r'^2 + r^2 - 2\mathbf{r}\cdot\mathbf{r}$$

$$\Delta^2 = r^2 + a'^2(1 - 2e'^2) + e'[2e'a'^2 + (2\mathbf{A}\cdot\mathbf{A}')(\cos E - e_0) + (2\mathbf{B}\cdot\mathbf{A}')\sin E] + (a'e')^2\cos^2 E'$$
$$- [2e'a'^2 + (2\mathbf{A}\cdot\mathbf{A}')(\cos E - e_0) + (2\mathbf{B}\cdot\mathbf{A}')\sin E]\cos E'$$
$$- [\qquad + (2\mathbf{A}\cdot\mathbf{B}')(\cos E - e_0) + (2\mathbf{B}\cdot\mathbf{B}')\sin E]\sin E'$$
$$= \gamma_0 + \gamma_2\cos^2 E' - \gamma_1\cos E' - \beta_0\sin E' \qquad \text{(Ausein. I: p. 139, (103))}$$
$$= H + w\cos^2 E' - K\cos\psi\cos E' - K\sin\psi\sin E' \qquad \text{(Astr. Papers Amer. Eph. V: p. 227)*}$$
$$= [C - q\cos(Q - E')][1 - q_1\cos(Q + E')] \qquad \text{(Ausein. I: p. 141, line 4)}$$
$$= [C - q\cos(E - E' + Q')][1 - q_1\cos(E + E' + Q')] \qquad ((8,8))$$

where $Q' = Q - E$. Expand as an identity in E' and compare coefficients.

$$C = H + \frac{w}{q^2}(q\sin Q)^2, \quad q\cos Q = \frac{K\cos\psi}{1 + C\,w/q^2}, \quad q\sin Q = \frac{K\sin\psi}{1 - C\,w/q^2}, \quad q_1 = \frac{w}{q}. \qquad ((8,9))$$

Then $$\Delta^{-2s} = [C - q\cos(E - E' + Q')]^{-s}[1 - q_1\cos(E + E' + Q')]^{-s}, \qquad ((8,10))$$
where $s = 1/2, 3/2$.

The numerical development of the first factor constitutes a considerable portion of the total computation, but it is greatly facilitated by Tables for the Development of the Disturbing Function given by Brown and Brouwer in the Transactions of the Yale Observatory, v. 6, pt. 5. Write

$$[C - q\cos(E - E' + Q')]^{-s} = k^{-s}[1 + A^2 - 2A\cos(E - E' + Q')]^{-s}$$
$$= k^{-s}(1 - A^2)^{-s}\left[\tfrac{1}{2}G_s^{(0)} + \sum_1^{\infty} G_s^{(j)} A^j \cos j(E - E' + Q')\right]$$
$$= \Sigma\, b_s^{(j)}\cos j(E - E' + Q') \qquad ((8,11))$$
$$= \Sigma\, \cos j(E - E')\, b_s^{(j)}\cos jQ' - \sin j(E - E')\, b_s^{(j)}\sin jQ'$$

where $C = k(1 + A^2)$, $q = 2Ak$, $A = \dfrac{q}{C + \sqrt{C^2 - q^2}}$, $k(1 - A^2) = \sqrt{C^2 - q^2}$, $(P,s) = -s\log\sqrt{C^2 - q^2}$
$\log b_s^{(j)} = \log G_s^{(j)} + j\log A + (P,s)$, and $b_s^{(0)}$ requires a coefficient of $\frac{1}{2}$. As this is the most convenient place to apply such constant factors as will eventually be needed, $(3m'a)$ in Δ^{-1} and $10m'a$ in Δ^{-3}, let

$$(P,1/2) = \log(3m'a) - \tfrac{1}{2}\log\sqrt{C^2 - q^2}, \quad (P,3/2) = (P,1/2) - \log\sqrt{C^2 - q^2} + (1 - \log 3). \qquad ((8,12))$$

These transformations have rendered the portions $b_s^{(j)}\cos jQ'$ and $b_s^{(j)}\sin jQ'$ entirely independent of E'; and through the functions H, $K\cos\psi$, $K\sin\psi$, C, q, Q, and Q', they depend upon the single variable E. Furthermore, this variable enters only through its cosine and sine, so that all these functions will be periodic. Therefore, let

$$b_s^{(j)}\cos jQ' = \Sigma\ (C^*,j,h)\cosh E + (S^*,j,h)\sinh E$$
$$b_s^{(j)}\sin jQ' = \Sigma\ (C^{*\prime},j,h)\cosh E + (S^{*\prime},j,h)\sinh E \qquad ((8,13))$$

Now by substituting n different values of E (usually distributed uniformly through 360°), each of these functions on the left may be evaluated n times and this will yield n linear equations from which n of the coefficients of the trigonometric series may be determined. This process is known as harmonic analysis, and it is most readily applied with 12, 16, or 24 different values of E, depending upon the rapidity of convergence along values of $\pm i$.

* At the time that these formulas were being reformulated from those given by Newcomb, the author was collaborating with Dr. S. Herrick, who suggested the use of the vectorial constants.

Finally (including the constant factors):

$$
\begin{aligned}
[C - q\cos(E - E' + Q')]^{-s} &= +\Sigma \cos j(E - E') \left\{ \Sigma (C^*,j,h) \cos hE + (S^*,j,h) \sin hE \right\} \\
&\quad - \Sigma \sin j(E - E') \left\{ \Sigma (C^{*\prime},j,h) \cos hE + (S^{*\prime},j,h) \sin hE \right\} \\
&= \Sigma\Sigma + \tfrac{1}{2}(C^*,j,h)\{+\cos([j+h]E - jE') + \cos([j-h]E - jE')\} \\
&\qquad + \tfrac{1}{2}(S^*,j,h)\{+\sin([j+h]E - jE') - \sin([j-h]E - jE')\} \\
&\qquad + \tfrac{1}{2}(C^{*\prime},j,h)\{-\sin([j+h]E - jE') - \sin([j-h]E - jE')\} \\
&\qquad + \tfrac{1}{2}(S^{*\prime},j,h)\{+\cos([j+h]E - jE') - \cos([j-h]E - jE')\} \\
&= \Sigma\Sigma\, (c,i,j) \cos(iE - jE') + (s,i,j) \sin(iE - jE') \qquad (8,14)
\end{aligned}
$$

where

$$
\begin{aligned}
(c,j+h,j) &= +\tfrac{1}{2}(C^*,j,h) + \tfrac{1}{2}(S^{*\prime},j,h) & (s,j+h,j) &= -\tfrac{1}{2}(C^{*\prime},j,h) + \tfrac{1}{2}(S^*,j,h) \\
(c,j-h,j) &= +\tfrac{1}{2}(C^*,j,h) - \tfrac{1}{2}(S^{*\prime},j,h) & (s,j-h,j) &= -\tfrac{1}{2}(C^{*\prime},j,h) - \tfrac{1}{2}(S^*,j,h) \\
(c,h,0) &= +\tfrac{1}{2}(C^*,0,h), \text{ including } h = 0, & (s,h,0) &= +\tfrac{1}{2}(S^*,0,h) \\
(c,j,j) &= +\ (C^*,j,0),\ j \neq 0, & (s,j,j) &= -\ (C^{*\prime},j,0),\ j \neq 0.
\end{aligned}
$$

The second factor may be treated in the same way, or it may be expanded by the binomial theorem as follows:

$$
\begin{aligned}
[1 - \tfrac{w}{q}\cos(E + E' + Q')]^{-s} &= 1 + s\frac{w}{q}\left\{\cos(-E - E')\cos Q' + \sin(-E - E')\sin Q'\right\} \\
&\quad + \frac{s(s+1)}{4}\left(\frac{w}{q}\right)^2 \left\{1 + \cos 2(-E - E')\cos 2Q' + \sin 2(-E - E')\sin 2Q'\right\} \\
&\quad + \ldots\ldots \\
&= 1 + \Sigma\Sigma\, (C,k,l)\cos(kE - lE') + (S,k,l)\sin(kE - lE'). \qquad (8,15)
\end{aligned}
$$

It is usually sufficient if $\frac{3}{2}\frac{w}{q}\cos Q'$ and $\frac{3}{2}\frac{w}{q}\sin Q'$ are obtained by harmonic analysis, and perhaps the constant term of $\frac{15}{16}\left(\frac{w}{q}\right)^2$. Then for $s = 1/2$, we simply divide by 3 all terms containing w as a factor, and divide by 5 all terms containing w^2 as a factor.

One further question remains to be considered before the actual computations may be begun: how many decimal places should be carried in each term? This resolves itself into two parts: what accuracy is wanted in the final results, and how many extra decimal places will be needed to protect the end figures of certain terms against the eventual small divisors which come in during the integrations? If only approximate results are desired, then five decimals of a radian should suffice for the final tables. For most cases, a limit of six decimal places is recommended, as this is almost certain to yield the maximum accuracy attainable with the first order perturbations only. If there is a possibility that the results will eventually be improved by the addition of second order perturbations, then it is advisable to carry through the original computations so as to obtain seven places in the final table. In the present example, the final tables will be given in units of the sixth decimal place, and all subsequent references to extra decimals shall be understood as applying to these units.

The second part of the question must, in general, be determined by an examination of the successive steps of the computation in their reverse order. Construct a table of $(i - jn'/n)$, as shown in the computations. Every small value, e.g. (1,2), (3,7), (4,9), will twice become the divisor of the coefficients in the Δ^{-1} series having the same indices (due to the constant term in M), and will once become the divisor of this and each of the adjoining coefficients above and below in both the Δ^{-1} and Δ^{-3} series (due to the (1,-1) terms in M and N). It is against these divisions that the protection of the end figures in these positions must be afforded. Due to the intermingling of terms during the various multiplications of series and transformations which must be performed, the adjoining positions, both vertically and horizontally, must be protected also, but to a somewhat lesser extent, as can be seen by an examination of the coefficients of the various multiplier series or the stencils which are actually used. Usually one less decimal place in each succeeding adjoining position is sufficient, except in the larger values of j, since the Bessel functions do not always converge with sufficient rapidity. The multiplication by i in forming $\partial/\partial E$ is partially

offset by a subsequent division by $(i - jn'/n)$ and is in most cases not troublesome. Finally, the terms (0,0) and (1,0) must be carried to the fullest possible accuracy, since they eventually become the secular terms. The number of extra places needed in each coefficient at the beginning of the work as a protection against some later operation is shown behind the divisors on Sheet 2 for the Δ^{-1} and Δ^{-3} series, respectively. Until the harmonic analysis is completed, it is necessary to carry for each separate value of j the maximum number of ~~extra~~ places required by any single term in that column, and these are shown at the bottom of each column.

The case of the term (4,9), in which $(i - jn'/n) = 0.0156$, requires special comment. This coefficient is of the 9th degree in A and the 5th degree in e, so that in general it would be very small, but after two integrations it is here magnified by a factor of nearly 5000. It is apparent that when the perturbations are based upon osculating elements, such a term is very sensitive to the epoch of osculation (which is itself arbitrary), for as the value of the osculating mean motion changes from one epoch to another, the effect of the corresponding change of the integrating divisor on such a critical term is

$$d(i - jn'/n)^{-2} = -\frac{2j}{(i - jn'/n)^3}\,\frac{n'}{n}\,\frac{dn}{n}.$$

In the present case, this has the value 2,100,000 dn/n. Within the limits of first order perturbations, the real significance of such a term becomes very questionable. The effect upon the motion of the planet, however, appears only in ndz and it is not serious. Let it be assumed that (c,4,9) and (s,4,9) in ndz contain numerical errors in the end figures due to inadequate protection against the small divisors of the double integration. This is the last step of the computation, except for the determination of the constants of integration, and so these errors are not distributed throughout any of the other computations. They are absorbed into the constant C at the epoch and they can have no effect upon ndz until there has been an appreciable change in the phase angle of these terms. But the phase angle changes very slowly, completing only one cycle in $1/(i - jn'/n)$ revolutions of the planet around the Sun. If, after a long interval of time, the effect of these errors begins to appear in the comparison with observations, then a correction to these terms and C may be determined. In the present example, these errors will be kept reasonably small by carrying three extra places in these terms at the beginning of the work.

The number of values of E needed for the harmonic analysis depends, ultimately, upon the satisfactory convergence of the series, and this is governed by the magnitude of e and the number of decimal places required. In the present example, twelve values of E (0°, 30°, 60°, ... 330°) will be used, and four additional values (45°, 135°, 225°, 315°) can always be used if they are needed for some terms of large j. The first quantities required in the calculation are given by the following formulas:

$$\begin{aligned}
r &= a - ae\cos E\\
K\cos\psi &= 2e'a'^2 - e_0(2\,\mathbf{A}\cdot\mathbf{A}') + (2\,\mathbf{A}\cdot\mathbf{A}')\cos E + (2\,\mathbf{B}\cdot\mathbf{A}')\sin E\\
K\sin\psi &= \qquad - e_0(2\,\mathbf{A}\cdot\mathbf{B}') + (2\,\mathbf{A}\cdot\mathbf{B}')\cos E + (2\,\mathbf{B}\cdot\mathbf{B}')\sin E\\
H &= a'^2(1 - 2e'^2) + r^2 + e'K\cos\psi
\end{aligned} \tag{(8,16)}$$

The numerical values in the present example are:

$$\begin{aligned}
r &= +3.02120603 - 0.28395663\cos E\\
K\cos\psi &= +1.4897894 + 11.944009\cos E + 28.914401\sin E\\
K\sin\psi &= +2.6816822 - 28.532225\cos E + 11.948236\sin E\\
H &= +26.943101 + r^2 + 0.0482538\,K\cos\psi
\end{aligned}$$

It is now apparent that a knowledge of vector analysis is not indispensable, since it enters the numerical application only in the formation of these simple "dot" products. The following check equations may be applied; n is the number of points of division of the circle and Σ signifies the sum of the n values of the same quantity. These checks should agree within a few units in the last place.

$$na - \Sigma r = 0,\quad n\,(2e'a'^2 - \{(2\,\mathbf{A}\cdot\mathbf{A}') + (2\,\mathbf{A}\cdot\mathbf{B}')\}e) - \Sigma K\cos\psi - \Sigma K\sin\psi = 0$$

$$n\,\{a'^2(1 - 2e'^2) + a^2(1 + \tfrac{1}{2}e^2)\} + e'\Sigma K\cos\psi - \Sigma H = 0.$$

The next set of formulas to be used is

$$C = H + \frac{w}{q^2}(q \sin Q)^2, \quad q \cos Q = \frac{K \cos \psi}{1 + C w/q^2}, \quad q \sin Q = \frac{K \sin \psi}{1 - C w/q^2} \qquad (8,17)$$

and these must be solved by the process of iteration. The solution for $E = 0$ is shown in detail, beginning with $q \cos Q = K \cos \psi$, $q \sin Q = K \sin \psi$, and working horizontally across each line. The following procedure is suggested. Form $(q \sin Q)^2$ and copy the result. Then add $(q \cos Q)^2$ and place the result, q^2, on the keyboard. Then derive w/q^2 by built-up division and copy. Clear the machine, set in H, then set w/q^2 on the keyboard and form C. Copy this and with the same keyboard setting, form $1 + C w/q^2$. Set this on the keyboard and derive $(q \cos Q)$ by built-up division and copy. Clear the product dials and multiply by -1. Set $1 - C w/q^2$ on the keyboard by reading from the product dials and as a check of this setting multiply again by -1. Then form $(q \sin Q)$ by built-up division, copy, and repeat the cycle until the solution converges to the final values.

$q \cos Q$	$q \sin Q$	$(q \sin Q)^2$	$0.0630287/q^2$	C
+13.433798	-25.850543	668.250573	0.0000742635	35.133494
+13.398839	-25.918167	671.751381	0.0000740399	35.133603
+13.398943	-25.917963	671.740806	0.0000740406	35.133603
+13.398943	-25.917964			

These solutions are tabulated on Sheet 3, along with other quantities which will be needed:

$$q^2, \quad q, \quad \sqrt{C^2 - q^2}, \quad \log\sqrt{C^2 - q^2}, \quad A = q/(C + \sqrt{C^2 - q^2}), \quad p = A/(1 - A^2), \quad \log A,$$

$$(P,1/2) = \log(3m'a) - \tfrac{1}{2}\log\sqrt{C^2 - q^2}, \quad (P,3/2) = (P,1/2) - \log\sqrt{C^2 - q^2} + (1 - \log 3),$$

$$\tan Q \text{ or } \cot Q, \quad Q' = Q - E, \quad \frac{3}{2}\frac{w}{q}, \quad \frac{3}{2}\frac{w}{q}\cos Q', \quad \frac{3}{2}\frac{w}{q}\sin Q',$$

where $\log(3m') = 7.4570273 - 10$, $\log(3m'a) = 7.9372077 - 10$, $(1 - \log 3) = 0.5228787$.

The determination of $\log \cos jQ'$ and $\log \sin jQ'$ requires no explanation. It is best to use a table of logarithmic trigonometric functions having the argument in decimals of a degree. The number of decimals required in the logarithms for each value of j may be gauged roughly (and it will not be underestimated) by first computing $\log b_s^{(j)}$ for the column in which A has the maximum value.

The computation of

$$\log b_s^{(j)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jQ' = \log G_s^{(j)} + j \log A + (P,s) + \log \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} jQ' \qquad (8,18)$$

is accomplished by means of the tables given in Transactions of the Yale Observatory, v. 6, pt. 5, and it will be found convenient to accumulate the results, including the interpolation of the tables, on the calculating machine. Both functions are computed together by adding the $\log \cos jQ'$ term last without clearing the keyboard. After this result is copied, subtract $\log \cos jQ'$ and then add $\log \sin jQ'$.

The next set of quantities is simply the antilogarithms of the quantities just computed. Since most of the work is with five or less significant figures, the Graphic Tables Combining Logarithms and Antilogarithms by Lacroix and Ragot (Mac Millan Co. New York, 1925) will be found useful.

The next step is the harmonic analysis of these natural values. This may be accomplished in several different ways. The formulas for most cases are given by Hansen, Ausein.I: p 159-164. A method of procedure is described by Encke in the 1857 Berliner Jahrbuch, and others by Brown in Transactions of the Yale Observatory, v. 6, p 62, 143. When a calculating machine with automatic division is available, the following scheme (for 12 points of division) will be found convenient. Arrange a table as shown below to correspond column for column with the natural values to be analyzed. Each line of the natural values is operated upon successively by each line of the table: simply multiply each natural value by the quantity in the same column directly below it, accumulate all of the products for that one line of the table and divide by the quantity shown at the right. The quotient is the value of the quantity shown at the left, and these should be arranged as shown at the top of Sheet 9. Then the quantities on Sheet 10 are obtained merely by addition or subtraction of adjoining values.

(C*,j,0)/2 + (C*,j,6)/2	+1	0	+1	0	+1	0	+1	0	+1	0	+1	0	12
(C*,j,0)/2 - (C*,j,6)/2	0	+1	0	+1	0	+1	0	+1	0	+1	0	+1	12
(C*,j,1)/4 + (C*,j,5)/4	+2	0	+1	0	-1	0	-2	0	-1	0	+1	0	24
(C*,j,1)/4 - (C*,j,5)/4	0	+1	0	0	0	-1	0	-1	0	0	0	+1	D
(C*,j,2)/4 + (C*,j,4)/4	+2	0	-1	0	-1	0	+2	0	-1	0	-1	0	24
(C*,j,2)/4 - (C*,j,4)/4	0	+1	0	-2	0	+1	0	+1	0	-2	0	+1	24
(C*,j,3)/2	+1	0	-1	0	+1	0	-1	0	+1	0	-1	0	12
(S*,j,3)/2	0	+1	0	-1	0	+1	0	-1	0	+1	0	-1	12
(S*,j,1)/4 + (S*,j,5)/4	0	+1	0	+2	0	+1	0	-1	0	-2	0	-1	24
(S*,j,1)/4 - (S*,j,5)/4	0	0	+1	0	+1	0	0	0	-1	0	-1	0	D
(S*,j,2)/4 + (S*,j,4)/4	0	+1	0	0	0	-1	0	+1	0	0	0	-1	D
(S*,j,2)/4 - (S*,j,4)/4	0	0	+1	0	-1	0	0	0	+1	0	-1	0	D

D = 13.8564065

For some planets of small mean motion and large eccentricity, it will be found practicable to use 12 points of division for all except a few terms of small i and large j. By carrying through parts of the computation for E = 45°, 135°, 225°, 315°, Brown's method (ibid. p.62) may be used. Otherwise for 16 or 24 uniformly distributed points of division, the arrangement is divided into two parts. If the 16 natural values are designated by (0), (1), (2), ... (15), and (0,8) = (0) + (8), (0/8) = (0) - (8), etc., prepare an intermediate sheet containing in the 16 columns the quantities (0,8), (1,9), . . . (7,15), (0/8), (1/9), . . . (7/15). Then the following table is applied, as explained above, to this intermediate sheet.

(C*,j,0)/2 + (C*,j,8)/2	+1		+1		+1		+1										16
(C*,j,0)/2 - (C*,j,8)/2		+1		+1		+1		+1									16
(C*,j,1)/4 + (C*,j,7)/4									+1+a		+1				-1		D
(C*,j,1)/4 - (C*,j,7)/4										+1		+a		-a		-1	d
(C*,j,2)/4 + (C*,j,6)/4	+1				-1												16
(C*,j,2)/4 - (C*,j,6)/4		+1		-1		-1		+1									D
(C*,j,3)/4 + (C*,j,5)/4									+1		-b				+b		16
(C*,j,3)/4 - (C*,j,5)/4										+a		-1		+1		-a	d
(C*,j,4)/2	+1		-1		+1		-1										16
(S*,j,4)/2		+1		-1		+1		-1									16
(S*,j,1)/4 + (S*,j,7)/4										+a		+1		+1		+a	d
(S*,j,1)/4 - (S*,j,7)/4											+b		+1		+b		16
(S*,j,2)/4 + (S*,j,6)/4		+1		+1		-1		-1									D
(S*,j,2)/4 - (S*,j,6)/4			+1				-1										16
(S*,j,3)/4 + (S*,j,5)/4										+1		-a		-a		+1	d
(S*,j,3)/4 - (S*,j,5)/4											+b		-1		+b		16

a = 0.41421356, b = 0.70710678, d = 17.3182752, D = 22.627417

If 24 or 32 points of division must be used, the scheme is even more complicated, but it may be arranged similarly. A simple method of checking this portion of the work is to synthesize the computed coefficients on Sheet 10 for E = 0°, 30°, 60°, and 90°, and thus reproduce the original natural values in these columns. The series on Sheet 11 are obtained from Sheet 10 by means of the small stencil #1. The constant term still requires a coefficient $\frac{1}{2}$, and this is now inserted as a separate factor.

Along with these operations, we must apply harmonic analysis to $\frac{3}{2}\frac{w}{q}\cos Q'$ and $\frac{3}{2}\frac{w}{q}\sin Q'$, but they are then combined by means of stencil #2. This portion of the work is shown at the bottom of Sheet 3, and the final results appear on the stencils #4 to #7.

Before the product $[C - q\cos(Q - E')]^{-s}[1 - q_1\cos(Q + E')]^{-s}$ can be formed, it will be necessary to consider the general problem of the multiplication of two double Fourier series. If one series (multiplicand) of the form

$$\Sigma\,\Sigma\ (c,i,j)\cos(iE - jE') + (s,i,j)\sin(iE - jE')$$

is to be multiplied by another series (multiplier) of the form

$$\Sigma\ \Sigma\ (C,k,l)\cos(kE - lE') + (S,k,l)\sin(kE - lE'),$$

the product is

$$\begin{aligned} P = 2\ \Sigma & + \tfrac{1}{2}(C,k,l)(c,i,j)\cos(iE - jE')\cos(kE - lE') \\ & + \tfrac{1}{2}(C,k,l)(s,i,j)\sin(iE - jE')\cos(kE - lE') \\ & + \tfrac{1}{2}(S,k,l)(c,i,j)\cos(iE - jE')\sin(kE - lE') \\ & + \tfrac{1}{2}(S,k,l)(s,i,j)\sin(iE - jE')\sin(kE - lE') \end{aligned} \qquad ((8,19))$$

where the limits of the summation indices remain the same as above. Then

$$\begin{aligned} P = \Sigma & + \tfrac{1}{2}(C,k,l)(c,i,j)\left[+\cos\{(i+k)E - (j+l)E'\} + \cos\{(i-k)E - (j-l)E'\}\right] \\ & + \tfrac{1}{2}(C,k,l)(s,i,j)\left[+\sin\{(i+k)E - (j+l)E'\} + \sin\{(i-k)E - (j-l)E'\}\right] \\ & + \tfrac{1}{2}(S,k,l)(c,i,j)\left[+\sin\{(i+k)E - (j+l)E'\} - \sin\{(i-k)E - (j-l)E'\}\right] \\ & + \tfrac{1}{2}(S,k,l)(s,i,j)\left[-\cos\{(i+k)E - (j+l)E'\} + \cos\{(i-k)E - (j-l)E'\}\right] \\ = \ & \Sigma\ \Sigma\ (\gamma,m,n)\cos(mE - nE') + (\sigma,m,n)\sin(mE - nE') \end{aligned} \qquad ((8,20))$$

where

$$\begin{aligned} (\gamma,m,n) = \Sigma & + \tfrac{1}{2}(C,k,l)(c,m-k,n-l) - \tfrac{1}{2}(S,k,l)(s,m-k,n-l) && m = i+k,\ n = j+l \\ & + \tfrac{1}{2}(C,k,l)(c,m+k,n+l) + \tfrac{1}{2}(S,k,l)(s,m+k,n+l) && m = i-k,\ n = j-l \\ (\sigma,m,n) = \Sigma & + \tfrac{1}{2}(S,k,l)(c,m-k,n-l) + \tfrac{1}{2}(C,k,l)(s,m-k,n-l) && m = i+k,\ n = j+l \\ & - \tfrac{1}{2}(S,k,l)(c,m+k,n+l) + \tfrac{1}{2}(C,k,l)(s,m+k,n+l) && m = i-k,\ n = j-l \end{aligned}$$

In order to apply these formulas by means of a simple, routine process, arrange the multiplicand series as follows:

.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
.....	(c,2,1)	-(s,2,1)	(c,2,0)	-(s,2,0)	(c,-2,1)	(s,-2,1)	(c,-2,2)	(s,-2,2)	.....
.....	(c,1,1)	-(s,1,1)	(c,1,0)	-(s,1,0)	(c,-1,1)	(s,-1,1)	(c,-1,2)	(s,-1,2)	.....
.....	(c,0,1)	-(s,0,1)	2(c,0,0)/2	0	(c,0,1)	(s,0,1)	(c,0,2)	(s,0,2)	.....
.....	(c,-1,1)	-(s,-1,1)	(c,1,0)	(s,1,0)	(c,1,1)	(s,1,1)	(c,1,2)	(s,1,2)	.....
.....	(c,-2,1)	-(s,-2,1)	(c,2,0)	(s,2,0)	(c,2,1)	(s,2,1)	(c,2,2)	(s,2,2)	.....
.....	.....	.....	.....	.....	.....	.....	.....	.....	.....

where the real series is written to the right of the broken line and the quantities to the left are merely an artifice to facilitate the routine computation. The artificial cosine terms are a "reflection" of the real cosine terms through the position 2(c,0,0)/2. The artificial sine terms are the negatives of the "reflection" of the real sine terms through the position (s,0,0) = 0. Arrange the multiplier series in exactly the same manner, but divide every term by 2, except the constant. The factor $\frac{1}{2}$ in the constant term 2(c,0,0)/2 is to be written explicitly and regarded as non-existent during the multiplication process, but it must be carried over and written with the new constant term of the product. Its purpose is merely to allow the application of the general procedure to the constant term without exception. Now superpose the multiplier series upon the multiplicand series so that (C,0,0) corresponds to any real value of (c,i,j), and form the sum of the products of all the corresponding superposed and subposed quantities, including the artificial or "reflected" quantities of both series. This sum is the value of (γ,i,j). The real portion of the multiplier series produces all the terms for which $m = i - k$, $n = j - l$, and the artificial portion produces the remaining terms for which $m = i + k$, $n = j + l$. The necessity for the artificial portion of the multiplicand series arises from the fact that if it were not used it would be necessary to form a number of terms with (C,0,0) corresponding to positions to the left of the broken line. They would then have to be transferred to positions to the right of the broken line in order to be within the limits defined by the summation signs above. This would be done by changing the sign of the angular argument and the sign of the sine terms. This, effectively; "reflects" these cosine and sine terms through their respective (0,0) positions, and is accomplished simultaneously with the other operations by means of the artificial portions of the multiplicand series.

Now rearrange the multiplier series so that each quantity is a "reflection" of the previous arrangement through the point midway between (C,0,0) and (S,0,0) = 0. Superpose this series upon the multiplicand series so that (C,0,0) corresponds to any real value of (s,i,j) and form all the corresponding products, as before. This is the value of (σ,i,j).

By this mechanical method, all the conditions of the formulas for (γ,m,n) and (σ,m,n) are satisfied. The application of the method will be seen in passing from Sheet 11 to Sheet 12 by means of the stencils #4 to #7. Since the multiplier series are, in the present application, all small, the terms of the subposed series will appear through the holes of the stencil and the terms with which they are to be multiplied are written adjacent to the holes. Otherwise, it would be necessary to double-space the series and cut the holes in the interspaces. The artificial or "reflected" portion of the multiplicand series should be written on a separate slip of paper and held in place by paper clips during the time they are needed for the multiplication operation. In general, since the constant terms are of the order of unity, the number of decimal places to be retained in each term of the product series is the same as the number there are in the term which multiplies (C,0,0).

The general form of the stencils for the multiplication by $[1 - q_1\cos(Q+E')]^{-S}$ is as follows:

Cos

..... $+\frac{1}{2}$(C,-2,1) $+\frac{1}{2}$(C,-1,1) 1 + (C,0,0) ..... $+\frac{1}{2}$(S,-2,1) $+\frac{1}{2}$(S,-1,1)

(c,i,j)

$+\frac{1}{2}$(C,-1,1) $+\frac{1}{2}$(C,-2,1) ..... $-\frac{1}{2}$(S,-1,1) $-\frac{1}{2}$(S,-2,1) .....

Sin

..... $-\frac{1}{2}$(S,-2,1) $-\frac{1}{2}$(S,-1,1) ..... $+\frac{1}{2}$(C,-2,1) $+\frac{1}{2}$(C,-1,1)

(s,i,j)

$+\frac{1}{2}$(S,-1,1) $+\frac{1}{2}$(S,-2,1) ..... 1 + (C,0,0) $+\frac{1}{2}$(C,-1,1) $+\frac{1}{2}$(C,-2,1) .....

Since $3m'a\Delta^{-1}$ is not needed explicitly, the series $(-3H)$ is prepared first and added to it to form $3a\Omega$, all in one operation.

$$3a\Omega = 3m'a\Delta^{-1} - 3m'a(xx' + yy' + zz')/r'^3 \qquad (8,21)$$

(Ausein. I: p. 64, 120)

Represent the second term by $(-3H) = -\frac{3m'a}{2a'^3}\left(\frac{a'}{r'}\right)^3 (2\mathbf{r}\cdot\mathbf{r}')$. To avoid a serious source of error, it seems advisable to eliminate subtraction operations whenever possible by transferring the minus sign to the formulas of the coefficients and then adding algebraically. Now

$$\begin{aligned}(2\mathbf{r}\cdot\mathbf{r}') = {} & 2ee'(2\mathbf{A}\cdot\mathbf{A}')/2\cos(0-0) \\ & - e'(2\mathbf{A}\cdot\mathbf{A}')\cos(+E-0) - e'(2\mathbf{B}\cdot\mathbf{A}')\sin(+E-0) \\ & + \tfrac{1}{2}\{+(2\mathbf{A}\cdot\mathbf{A}') - (2\mathbf{B}\cdot\mathbf{B}')\}\cos(-E-E') + \tfrac{1}{2}\{-(2\mathbf{B}\cdot\mathbf{A}') - (2\mathbf{A}\cdot\mathbf{B}')\}\sin(-E-E') \\ & - e(2\mathbf{A}\cdot\mathbf{A}')\cos(0-E') + e(2\mathbf{A}\cdot\mathbf{B}')\sin(0-E') \\ & + \tfrac{1}{2}\{+(2\mathbf{A}\cdot\mathbf{A}') + (2\mathbf{B}\cdot\mathbf{B}')\}\cos(+E-E') + \tfrac{1}{2}\{+(2\mathbf{B}\cdot\mathbf{A}') - (2\mathbf{A}\cdot\mathbf{B}')\}\sin(+E-E')\end{aligned}$$

and

$$\left(\frac{a'}{r'}\right)^3 = \Sigma\,(c,j)\cos jE' \qquad (8,22)$$

which may be obtained by harmonic analysis. Using $e' = 0.04825382$, the values of (c,0) to (c,5) in units of the 8th decimal place are: 100701590, 14560756, 702612, 28262, 1024, 36. If $(2\mathbf{r}\cdot\mathbf{r}')$ is written out explicitly, as shown on Sheet 1, then $(-3H)$ may be found by means of stencil #3, consisting merely of a straight edge, where the numerical values to be used for the coefficients are $-a10^{-11}$ multiplied by 1024051, 74035, 3572, 144, 5, respectively. The arrangement of this stencil is

..... $\frac{1}{2}$(c,2) $\frac{1}{2}$(c,1) (c,0) $\frac{1}{2}$(c,1) $\frac{1}{2}$(c,2) .....

Since this multiplier series consists of cosines only, the stencil for the formation of the sine and cosine terms of the product is the same. The result is $(-3H)$, and this is placed at the bottom of Sheet 12. As the individual terms of $3m'a\Delta^{-1}$ are obtained, those of $(-3H)$ are added in their proper positions and the result is $3a\Omega$, as shown at the top of Sheet 12.

The first series on Sheet 13 is obtained very easily. If

$$3a\Omega = \Sigma\ (c,i,j)\cos(iE - jE') + (s,i,j)\sin(iE - jE')$$

then
$$\frac{\partial(3a\Omega)}{\partial E} = \Sigma\ i(s,i,j)\cos(iE - jE') - i(c,i,j)\sin(iE - jE') \tag{(8,23)}$$

The second series is obtained from

$$ar\frac{\partial\Omega}{\partial r} = \tfrac{1}{2}(r'^2 - r^2)\, m'a\Delta^{-3} - \tfrac{1}{2}m'a\Delta^{-1} + (H) \tag{(8,24)}$$

(Ausein. I: p. 119)

This may be reduced to

$$ar\frac{\partial\Omega}{\partial r} = \frac{(r'^2 - r^2)}{20}\, 10\,m'a\Delta^{-3} - \frac{1}{6}(3a\Omega) + \frac{1}{2}(-3H) \tag{(8,25)}$$

The general form of this stencil #8, which is applied to Sheet 12 to obtain $ar\frac{\partial\Omega}{\partial r}$, is

$\boxed{(c,i,j)}$ -0.16667

			$+\frac{1}{2}(C,2,0)$		
			$+\frac{1}{2}(C,1,0)$		
		(c,i,j)	$+ (C,0,0)$		
$+\frac{1}{2}(C,0,2)$	$+\frac{1}{2}(C,0,1)$		$+\frac{1}{2}(C,1,0)$	$+\frac{1}{2}(C,0,1)$	$+\frac{1}{2}(C,0,2)$
			$+\frac{1}{2}(C,2,0)$		

$\boxed{(c,i,j)}$ +0.5

The upper and lower holes in this stencil must correspond to the same indices in the $(3a\Omega)$ and $(-3H)$ series that (C,0,0) does in the $10\,m'a\Delta^{-3}$ series. Again, this multiplier contains only cosines, so the same stencil serves for both cosine and sine terms of the product. The formulas for the coefficients are:

$$(C,0,0) = \frac{a'^2(1 + \frac{1}{2}e'^2) - a^2(1 + \frac{1}{2}e^2)}{20} \tag{(8,26)}$$

$$\tfrac{1}{2}(C,1,0) = \frac{a^2e}{20},\quad \tfrac{1}{2}(C,2,0) = -\frac{(ae)^2}{80},\quad \tfrac{1}{2}(C,0,1) = -\frac{a'^2e'}{20},\quad \tfrac{1}{2}(C,0,2) = \frac{(a'e')^2}{80}.$$

The actual values in the present case are shown on stencil #8. The purpose of the factor 10 in $10\,m'a\Delta^{-3}$ is to bring (C,0,0) to the order of unity.

The next two steps involve the transformation of the eccentric anomaly of the disturbing planet to a variable that changes linearly with the independent variable, which is the eccentric anomaly of the disturbed planet. We eliminate E' in terms of the mean anomaly, g', by means of the transformation

$$\begin{matrix}\cos\\ \sin\end{matrix}(iE - jE') = \sum_{k=-\infty}^{\infty} P_k^{(j)} \begin{matrix}\cos\\ \sin\end{matrix}(iE - kg') \tag{(8,27)}$$

where
$$P_k^{(j)} = \frac{j}{k} J(k-j, \tfrac{1}{2}e'k).$$

(Ausein. I: p. 170)

This may be reduced to a routine process, similar to the multiplication process, by using the stencils #9. These are essentially a collection of straight edge stencils, one for each different value of j. The "reflected" portion of the series to be transformed must be attached, and then, if the central heavy line is placed under any real quantity (c,i,j) or (s,i,j), the corresponding new term in the transformed series is the sum of the products of all the terms along the straight edge multiplied by the corresponding quantities directly below and in the appropriate horizontal line,

depending upon the value of j indicated at the left. The numerical quantities on this stencil are based upon the value $e' = 0.04825382$ and, since they are independent of the individual minor planet, they may be used repeatedly for any case. The transformed series are shown on Sheet 14.

The elimination of g' in terms of $\phi = \frac{n'}{n} E - \frac{n'}{n} g_0 + g'_0$ is accomplished by means of the transformation

$$\begin{aligned} ((c,i,j)) &= (c,i,j)\, J(0,jx) + (c,i-1,j)\, J(1,jx) + (c,i-2,j)\, J(2,jx) + \ldots \\ &\quad - (c,i+1,j)\, J(1,jx) + (c,i+2,j)\, J(2,jx) - \ldots \end{aligned} \qquad ((8,28))$$

(Ausein. I: p. 181)

and similarly for the sine terms, where $x = \frac{1}{2} e n'/n$. The transformed series on Sheet 15 are obtained from those on Sheet 14 by means of the stencils #10. These are applied in the same manner as stencils #9, except that the straight edge is vertical instead of horizontal.

There are two principal methods of computing these Bessel functions, J(k,x). The first is by direct application of the formula

$$J(k,x) = x^k \left[\frac{1}{k!} - \frac{x^2/1!}{(k+1)!} + \frac{x^4/2!}{(k+2)!} - \frac{x^6/3!}{(k+3)!} + \ldots \right] \qquad ((8,29))$$

If the values of the quantities in the numerators are written in a vertical column and the values of 1/k! are written to correspond line for line on a slip of paper which may be slid vertically, then each required value of the square bracket may be obtained as the sum of the products of the adjacent quantities for the appropriate position of the slip of paper.

The second method depends upon the formula

$$J(k,x) = J(0,x)\, p_1 p_2 p_3 \cdots p_k, \qquad p_k = \frac{1}{k/x - p_{k+1}} \qquad ((8,30))$$

(Ausein. I: p. 172,173)

Assume some $p_{k+1} = 0$ and find all the p's of lower subscript. As a check, repeat by assuming that $p_{k+2} = 0$. If these do not agree for a sufficiently large value of k, then a larger value of k should have been used in the first place. This method is better when a large number of Bessel functions for the same argument are required.

The third component of the perturbative function is

$$a^2 \frac{\partial \Omega}{\partial Z} = m' a (\Delta^{-3} - r'^{-3})\, Z' \qquad ((8,31))$$

(Ausein. I: p. 106)

A geometrical interpretation of the expression given by Hansen shows that

$$Z' = \mathbf{C} \cdot \mathbf{r}' = (\mathbf{C} \cdot \mathbf{A}')(\cos E' - e') + (\mathbf{C} \cdot \mathbf{B}') \sin E' \qquad ((8,32))$$

Now the series $10\, m' a \Delta^{-3}$ is already known, and since $- 10\, m' a r'^{-3}$ consists of only a few cosine terms, these may be easily added without rewriting the entire series. The values of the quantities to be added to 2(c,0,0)/2, (c,0,1), (c,0,2), ... of $10\, m' a \Delta^{-3}$ are $- a\, 10^{-11}$ multiplied by 13654013/2, 987138, 47633, 1916, 69, respectively. These new terms are shown at the bottom of Sheet 12. The multiplication by $Z'/10$ is accomplished by stencils #11 and #12, the general form of which is:

Cos:	$+\frac{1}{2}(C,0,1)$	$-\frac{1}{2}(S,0,1)$	$\overline{(C,0,0)}$	0	$+\frac{1}{2}(C,0,1)$	$+\frac{1}{2}(S,0,1)$
Sin:	$+\frac{1}{2}(S,0,1)$	$+\frac{1}{2}(C,0,1)$	0	$\overline{(C,0,0)}$	$-\frac{1}{2}(S,0,1)$	$+\frac{1}{2}(C,0,1)$

where $(C,0,0) = - e'(\mathbf{C} \cdot \mathbf{A}')/10$, $\frac{1}{2}(C,0,1) = (\mathbf{C} \cdot \mathbf{A}')/20$, $\frac{1}{2}(S,0,1) = - (\mathbf{C} \cdot \mathbf{B}')/20$. The results are given on Sheet 16. The transformation from E' to g' and g' to ϕ are obtained in exactly the same manner as before, and these results are shown on Sheet 17.

The next operation involves the series M, N, and Q, which have as their angular arguments $(hH + kE)$. The angle H has a curious role, but really serves a very ingenious purpose. This has been described in various ways, the most understandable of which is the exposition given by G. W. Hill in his Collected Works, v. 1, p. 155. The angle H is simply an E which is to be regarded

as a constant during an integration with respect to E. After the integration, the function of H and E is to be "dashed" (indicated by the operator $\overline{\quad}$), i.e E is written in place of H.

At this point we see two distinct advantages of the eccentric anomaly as independent variable, namely that h = -1, 0, +1 only, and that M, N, and Q are finite series. If any other independent variable were used, these indices and series would extend to infinity. The coefficients of these series are:

		M	N	Q	M	N	Q
h	k	cos	sin	sin	cos	sin	sin
0	0	$-3(1-\frac{1}{2}e^2)/(1-e^2)$	0	0	$-1-v$	0	0
0	1	$2e/($ " $)$	$e/(1-e^2)$	e	$+2/3\,u$	$+u$	$+\frac{1}{2}e$
0	2	$-\frac{1}{2}e^2/($ " $)$	$-\frac{1}{2}e^2/($ " $)$	$-\frac{1}{2}e^2$	$-1/6\,v$	$-\frac{1}{2}v$	$-(\frac{1}{2}e)^2$
1	1	$e^2/($ " $)$	$e^2/($ " $)$	$+\frac{1}{2}e^2$	$+1/3\,v$	$+v$	$+(\frac{1}{2}e)^2$
1	0	$-3e/($ " $)$	$-e/($ " $)$	$-1.5\,e$	$-u$	$-u$	$-3/4\,e$
1	-1	$(4-e^2)/($ " $)$	$-(2-e^2)/($ " $)$	$1+\frac{1}{2}e^2$	$+2/3+v$	$-1-v$	$+\frac{1}{2}+(\frac{1}{2}e)^2$
1	-2	$-e/($ " $)$	$e/($ " $)$	$-\frac{1}{2}e$	$-1/3\,u$	$+u$	$-1/4\,e$

The actual quantities to be used on the stencils may be obtained from the formulas shown at the right, in terms of the auxiliaries: $u = \frac{1}{2}e/(1-e^2)$, $v = eu$. The purpose of the factor 3 in $3a\Omega$ is to bring (C,0,0) of M to the order of unity.

The formation of
$$T = \frac{1}{3} M \frac{\partial(3a\Omega)}{\partial E} + Nar\frac{\partial\Omega}{\partial r} \qquad ((8,33))$$

must be accomplished by means of a multiplication similar to ((8,19)), but including hH in the arguments. This result may also be integrated, all in one operation, to give $W = \int T\,dE$. The details will not be developed, but if

$$W = \Sigma\ (C,h,i,j)\cos(hH + iE - j\phi) + (S,h,i,j)\sin(hH + iE - j\phi)$$

then

$$\begin{aligned}
(C,-1,i,j) &= \Sigma\ \{-\tfrac{1}{2}(C,1,k)(s,i+k,j) + \tfrac{1}{2}(S,1,k)(c,i+k,j)\} \div (i - jn'/n)\\
(C,0,i,j) &= \Sigma\ \{-\tfrac{1}{2}(C,0,k)(s,i-k,j) - \tfrac{1}{2}(S,0,k)(c,i-k,j)\\
&\qquad -\tfrac{1}{2}(C,0,k)(s,i+k,j) + \tfrac{1}{2}(S,0,k)(c,i+k,j)\} \div (i - jn'/n)\\
(C,1,i,j) &= \Sigma\ \{-\tfrac{1}{2}(C,1,k)(s,i-k,j) - \tfrac{1}{2}(S,1,k)(c,i-k,j)\} \div (i - jn'/n)\\
(S,-1,i,j) &= \Sigma\ \{+\tfrac{1}{2}(C,1,k)(c,i+k,j) + \tfrac{1}{2}(S,1,k)(s,i+k,j)\} \div (i - jn'/n)\\
(S,0,i,j) &= \Sigma\ \{+\tfrac{1}{2}(C,0,k)(c,i-k,j) - \tfrac{1}{2}(S,0,k)(s,i-k,j)\\
&\qquad +\tfrac{1}{2}(C,0,k)(c,i+k,j) + \tfrac{1}{2}(S,0,k)(s,i+k,j)\} \div (i - jn'/n)\\
(S,1,i,j) &= \Sigma\ \{+\tfrac{1}{2}(C,1,k)(c,i-k,j) - \tfrac{1}{2}(S,1,k)(s,i-k,j)\} \div (i - jn'/n)
\end{aligned} \qquad ((8,34))$$

except when the divisor is zero. In these cases, place the undivided, accumulated products in the corresponding position (i.e. same indices) of the co-function, include a factor $\frac{1}{2}E$, and change the signs of the quantities which have been placed in the sine column.

The general forms of the stencils which are to be applied to Sheet 15 to obtain the first six columns of Sheet 18 by the formal multiplication process are shown at the top of the next page. Sheet 2 must be attached at the bottom of Sheet 15 with paper clips so that $\div$ always indicates the divisor $(i - jn'/n)$ corresponding to the same indices (i,j) indicated by the heavy mark in both multiplicand series above. The artificial, reflected portions of these latter series must not be neglected in the j = 0 column. All six stencils must be applied to all the real positions of the (i,j) indices. In practice only three stencil sheets are necessary, since the cosine and sine terms for the same value of h may be placed on opposite sides of the same stencil. The resulting terms of W (and R) are all given uniformly to one extra decimal place as a protection of the end figures of $d\nu/dE$ (and du/dE), except for the secular terms.

The seventh and eighth columns give $\overline{W} = d(ndz)/dnt$, and these are obtained by simple addition of three diagonally adjacent quantities in the first six columns with the aid of stencil #19. It is necessary to write 2(c,-1,1,0)/2 = 2(c,0,0)/2 and (s,-1,1,0) = (s,0,0) = 0, regardless of the numerical value which may be obtained by the formal process.

(C,-1,i,j)

cos	sin
$-\frac{1}{2}$(C,1,-2) $-\frac{1}{2}$(C,1,-1) $-\frac{1}{2}$(C,1,0) $-\frac{1}{2}$(C,1,1)	(s,i,j)
(c,i,j)	$+\frac{1}{2}$(S,1,-2) $+\frac{1}{2}$(S,1,-1) $+\frac{1}{2}$(S,1,0) $+\frac{1}{2}$(S,1,1)

÷

(C,0,i,j)

cos	sin
$-\frac{1}{2}$(C,0,2) $-\frac{1}{2}$(C,0,1) $-$ (C,0,0) $-\frac{1}{2}$(C,0,1) $-\frac{1}{2}$(C,0,2)	(s,i,j)
(c,i,j)	$-\frac{1}{2}$(S,0,2) $-\frac{1}{2}$(S,0,1) 0 $+\frac{1}{2}$(S,0,1) $+\frac{1}{2}$(S,0,2)

÷

(C,1,i,j)

cos	sin
$-\frac{1}{2}$(C,1,1) $-\frac{1}{2}$(C,1,0) $-\frac{1}{2}$(C,1,-1) $-\frac{1}{2}$(C,1,-2)	(s,i,j)
(c,i,j)	$-\frac{1}{2}$(S,1,1) $-\frac{1}{2}$(S,1,0) $-\frac{1}{2}$(S,1,-1) $-\frac{1}{2}$(S,1,-2)

÷

(S,-1,i,j)

cos	sin
(c,i,j)	$+\frac{1}{2}$(C,1,-2) $+\frac{1}{2}$(C,1,-1) $+\frac{1}{2}$(C,1,0) $+\frac{1}{2}$(C,1,1)
$+\frac{1}{2}$(S,1,-2) $+\frac{1}{2}$(S,1,-1) $+\frac{1}{2}$(S,1,0) $+\frac{1}{2}$(S,1,1)	(s,i,j)

÷

(S,0,i,j)

cos	sin
(c,i,j)	$+\frac{1}{2}$(C,0,2) $+\frac{1}{2}$(C,0,1) $+$ (C,0,0) $+\frac{1}{2}$(C,0,1) $+\frac{1}{2}$(C,0,2)
$-\frac{1}{2}$(S,0,2) $-\frac{1}{2}$(S,0,1) 0 $+\frac{1}{2}$(S,0,1) $+\frac{1}{2}$(S,0,2)	(s,i,j)

÷

(S,1,i,j)

cos	sin
(c,i,j)	$+\frac{1}{2}$(C,1,1) $+\frac{1}{2}$(C,1,0) $+\frac{1}{2}$(C,1,-1) $+\frac{1}{2}$(C,1,-2)
$-\frac{1}{2}$(S,1,1) $-\frac{1}{2}$(S,1,0) $-\frac{1}{2}$(S,1,-1) $-\frac{1}{2}$(S,1,-2)	(s,i,j)

÷

The ninth and tenth columns give $n\,dz = \int \overline{W}\,(1 - e\cos E)\,dE$. The multiplication and the integration may again be performed in a single step by applying the stencils #20 and #21 to the sine and cosine columns, respectively, of $\overline{W}$ to obtain the cosine and sine columns, respectively, of $n\,dz$. In every case, there must be a final division by $(i - jn'/n)$. The secular terms introduce some added complexities which are described by the following formulas:

If
$$\begin{aligned}\overline{W} = {} & 2(c,0,0)/2 + 2(c',0,0)E/2 + (c,1,0)\cos E + (c',1,0)E\cos E + (c,2,0)\cos E + \ldots \\ & + (s,1,0)\sin E + (s',1,0)E\sin E + (s,2,0)\sin E + \ldots\end{aligned} \tag{8,35}$$

then
$$\begin{aligned} n\,dz = {} & \{2(c,0,0) - 2(c,1,0)e/2\}E/2 + \{(c',1,0) - 2(c',0,0)e/2\}(E\sin E + \cos E) + \\ & (s',1,0)(-E\cos E + \sin E) - (c',1,0)e/4\,(E\sin 2E + \tfrac{1}{2}\cos 2E) - (s',1,0)e/4\,(-E\cos 2E + \tfrac{1}{2}\sin 2E) \\ & + \{-(s,1,0) + \tfrac{1}{2}e\,(s,2,0)\}\cos E + \{-\tfrac{1}{2}e\,2(c,0,0) + (c,1,0) - \tfrac{1}{2}e\,(c,2,0)\}\sin E \\ & + \tfrac{1}{2}\{+\tfrac{1}{2}e\,(s,1,0) - (s,2,0) + \tfrac{1}{2}e\,(s,3,0)\}\cos 2E + \tfrac{1}{2}\{-\tfrac{1}{2}e\,(c,1,0) + (c,2,0) - \tfrac{1}{2}e\,(c,3,0)\}\sin 2E \\ & + \ldots\end{aligned} \tag{8,36}$$

There is also a term $\{2(c',0,0) - 2(c',1,0)e/2\}E^2/4$, but this is identically zero.

The eleventh to fourteenth columns contain ν and $d\nu/dE$. which may be obtained by two independent methods. Firstly, if we use the formula which Hansen gives: $\frac{d\nu}{dE} = -\frac{1}{2}\overline{\left(\frac{\partial W}{\partial H}\right)}$. Since

$$W = \Sigma\ (C,h,i,j)\cos(hH + iE - j\phi) + (S,h,i,j)\sin(hH + iE - j\phi),$$

$$\frac{d\nu}{dE} = +\tfrac{1}{2}\,\Sigma\ \{(S,-1,i+1,j) - (S,1,i-1,j)\}\cos(iE - j\phi) + \{-(C,-1,i+1,j) + (C,1,i-1,j)\}\sin(iE - j\phi).$$

A stencil cut similar to #19 may be used for this purpose, but instead of all the multipliers being +1, they will be $+\frac{1}{2}$, 0, $-\frac{1}{2}$ for the cosine terms and $-\frac{1}{2}$, 0, $+\frac{1}{2}$ for the sine terms of $d\nu/dE$. These are also to be applied to the secular terms without exception. Then to obtain ν by integration, the procedure is the same as described for ndz above, except that it is necessary to put $e = 0$ in the latter formulas.

Secondly, $$\nu = -(X_0 + eX_1)/6 - \tfrac{1}{2}\,\overline{W} \qquad ((8,37))$$

This is the formula given by G. W. Hill in his Collected Works, v. 1, p. 349. The stencil for this operation is #22, giving:

$$(c,i,j) = -(C,-1,i,j)\,e/6 - (C,0,i,j)/6 - (C,1,i,j)\,e/6 - (c,i,j)/2$$

and similarly for the sine. Notice that $2(c,0,0)/2$ and $(c',0,0)E/2$ of the ν series are equal to the terms $-(C,-1,1,0)$ and $-(C',0,0,0)E/2$, respectively, of the W series.

The thirteenth and fourteenth columns are derived from the eleventh and twelfth by direct differentiation with respect to E. If ν is of the same form as ((8,35)) above, then

$$\begin{aligned}\frac{d\nu}{dE} = 2(c',0,0)/2 &+ [(s,1,0) + (c',1,0)]\cos E + (s',1,0)E\cos E + 2(s,2,0)\cos E + \dots \\ &- [(c,1,0) - (s',1,0)]\sin E - (c',1,0)E\sin E - 2(c,2,0)\sin E + \dots\end{aligned}$$

The former method has the disadvantages that the value of the constant of integration is not given directly, and the terms of ν are poorly determined whenever $(i - jn'/n)$ is small. The latter method has the disadvantage that the terms of $\frac{d\nu}{dE}$ are poorly determined whenever $(i - jn'/n)$ is large, unless extra decimal places are carried in ν. The most reasonable procedure would therefore be to use the first method, and then check ν by means of the second method, giving preference to the values of the ν terms computed by the second method whenever $(i - jn'/n)$ is small. This also provides a check on $\overline{W}$. The second method can also be used to determine the constant term. In the illustration, the second method was used, and the first method was used to check $d\nu/dE$.

To obtain R in columns 15 to 20, it is necessary to apply the stencils #13 to #18 to the lower half of Sheet 17. This time the coefficients of the N series are replaced by those which come from the Q series and the upper part of the stencil is left blank. Attach Sheet 2 in place below Sheet 17 to indicate the appropriate divisor. Then columns 21 and 22 are obtained by means of stencil #19 (similar to columns 7 and 8); and columns 23 and 24 are obtained by differentiation (similar to the columns 13 and 14).

The factor E in the secular terms may now be replaced by $nt + e\sin E$. If any one of our series is of the form:

$$\begin{aligned}2(c,0,0)/2 + 2(c',0,0)E/2 + (c,1,0)\cos E + (c',1,0)E\cos E + (c,2,0)\cos 2E + (c',2,0)E\cos 2E + \dots \\ + (s,1,0)\sin E + (s',1,0)E\sin E + (s,2,0)\sin 2E + (s',2,0)E\sin 2E + \dots\end{aligned}$$

then replace E by nt (where n is the mean motion in radians per unit of t), and add the following terms (which are due to $e\sin E$):

$$\begin{aligned}&(s',1,0)e/2 + (s',2,0)e/2\,\cos E - (s',1,0)e/2\cos 2E - (s',2,0)e/2\cos 3E \\ &+ \{2(c',0,0)e/2 - (c',2,0)e/2\}\sin E + (c',1,0)e/2\,\sin 2E + (c',2,0)e/2\,\sin 3E.\end{aligned}$$

The result of this substitution is given at the bottom of Sheet 18. In the present illustration, $nt = 32.7576\,T$, where $T = 0.0001\,(t - t_0)^d$.

The series which determine the three components of the perturbation are then complete, except for the constants of integration. The most general function which may be added to W, but which are independent of E are $k_0 + k_1\cos H + k_2\sin H$. In $\overline{W}$, these become $k_0 + k_1\cos E + k_2\sin E$. After the second integration, the constants in ndz, ν, $d\nu/dE$, u, and du/dE are:

$$(C - g_0) + (k_0 - \tfrac{1}{2}ek_1)nt + (1 - \tfrac{1}{2}e^2)k_1\sin E - k_1e/4\sin 2E - k_2\cos E + k_2e/4\cos 2E,$$

$$-\frac{2}{3}k_0 - \frac{e}{6}k_1 - \tfrac{1}{2}\cos E\,k_1 - \tfrac{1}{2}\sin E\,k_2 \qquad \tfrac{1}{2}\sin E\,k_1 - \tfrac{1}{2}\cos E\,k_2,$$

$$l_1(\cos E - e) + l_2\sin E, \qquad -l_1\sin E + l_2\cos E.$$

Since the fixed elliptic orbit is the osculating orbit at the epoch, the constants of integration must satisfy the conditions that the perturbations and their first derivatives vanish at the epoch. These conditions are expressed by the equations (Ausein. II: p. 94):

$$\begin{array}{llllll}
(C - g_0) + & \left[(1 - \tfrac{1}{2}e^2)\sin E - \tfrac{e}{4}\sin 2E\right] k_1 & - \left[\cos E - \tfrac{e}{4}\cos 2E\right] k_2 + & n\,dz & = 0 & \\
k_0 & + \cos E\, k_1 & + \sin E\, k_2 + & \overline{W} & = 0 & \\
- k_0 & - \left[\tfrac{e}{4} + \tfrac{3}{4}\cos E\right] k_1 & - \tfrac{3}{4}\sin E\, k_2 + & \tfrac{3}{2}\nu & = 0 & \\
 & + \sin E\, k_1 & - \cos E\, k_2 + & 2\,\dfrac{d\nu}{dE} & = 0 & \\
 & + (\cos E - e)\, l_1 & + \sin E\, l_2 + & u & = 0 & \\
 & - \sin E\, l_1 & + \cos E\, l_2 + & \dfrac{du}{dE} & = 0 &
\end{array} \qquad ((8,38))$$

Eliminating k_0 from the second and third equations gives

$$+ (\cos E - e)\, k_1 + \sin E\, k_2 = - 4\,\overline{W} - 6\nu$$
$$- \sin E\, k_1 + \cos E\, k_2 = + 2\,\frac{d\nu}{dE}$$

Thus k_1 and k_2 are determined with the same set of coefficients as l_1 and l_2; and then $(C - g_0)$ and k_0 are each determined separately. The six series must all be evaluated at the epoch, with their constants of integration and secular terms all set equal to zero.

To evaluate the series at the epoch, let each of the arguments be expressed in the form

$$iE - j\phi = (i - j\,n'/n)(E - E_0) + [\,iE_0 - j\{g_0' + (E_0 - g_0)\,n'/n\}\,] = a(E - E_0) + b \qquad ((8,39))$$

and find b for all necessary values of (i,j). In all these phase angles, not only here, but also later in the comparison of the series with observations, it is advisable to measure the angles in units of 2π radians (also called "circles" or "gones"), so that only the decimal part of the angle is needed to determine the cosine and sine; the number of whole revolutions or cycles is indicated by the integer before the decimal point. By placing the phase angles and their sines and cosines to correspond line for line with their appropriate coefficients, as on the extreme righthand side of Sheet 18, the series may be evaluated by accumulating the sums of the products of all the proper pairs of factors. The complete series in their final form are shown on Sheet 19. As far as first order perturbations are concerned, the constant term of ndz may be added to the mean anomaly at the epoch, and the secular term may be added to the mean motion; this has been done in the final series.

Sheet 1: Epoch of Osculation = 1935 July 17.0 UT = JD 2428000.5

A	**B**	**C**	**A'**	**B'**
+1.76017823	+2.43748948	-0.18748727	+5.061850	-1.195743
-2.40311102	+1.66281027	-0.75044183	+1.145326	+4.636851
-0.50449567	+0.58374783	+2.92050875	+0.367512	+2.018895

a^2	9.12768586	e	0.09398784
a	3.02120603	e	5°.385107
P	5.25134455	n	0.18768728

a'	5.202803	e'	0.0482538
n'	0°.08309122	$(a'e')^2$	0.0630287
$a'^2(1 + \frac{1}{2}e'^2)$	27.100672	$2e'a'^2$	2.612381
$a'^2(1 - 2e'^2)$	26.943101		

(2 **A**·**A'**)	+11.944009	(2 **A**·**B'**)	-28.532225
(2 **B**·**A'**)	+28.914401	(2 **B**·**B'**)	+11.948236
(**C**·**A'**)	- 0.735211	(**C**·**B'**)	+ 2.640700

g_0'	$0^c.6222505$
g_0	0.4575692
E	0.4611816
cos E	-0.9704028
(cos E − e)	-1.0643906
sin E	

2 **r**·**r'**

j =	0		1	
i	cos	sin	cos	sin
-1			-0.0021135	-0.1910880
0	+0.1083386/2	0.0	-1.1225916	-2.6816822
1	-0.5763438	-1.3952297	+11:9461225	+28.7233130

E	cos E		sin E		--
0.000	1.00 000	2	0.00 000	628	0.250
0.001	0.99 998	6	0.00 628	629	0.249
0.002	0.99 992	10	0.01 257	628	0.248
0.003	0.99 982	14	0.01 885	628	0.247
0.004	0.99 968	17	0.02 513	628	0.246
0.005	0.99 951	22	0.03 141	628	0.245
0.006	0.99 929	26	0.03 769	628	0.244
0.007	0.99 903	29	0.04 397	627	0.243
0.008	0.99 874	34	0.05 024	628	0.242
0.009	0.99 840	37	0.05 652	627	0.241
0.010	0.99 803	42	0.06 279	627	0.240
0.011	0.99 761	45	0.06 906	627	0.239
0.012	0.99 716	49	0.07 533	626	0.238
0.013	0.99 667	54	0.08 159	626	0.237
0.014	0.99 613	57	0.08 785	626	0.236
0.015	0.99 556	61	0.09 411	625	0.235
0.016	0.99 495	65	0.10 036	625	0.234
0.017	0.99 430	69	0.10 661	625	0.233
0.018	0.99 361	73	0.11 286	624	0.232
0.019	0.99 288	77	0.11 910	623	0.231
0.020	0.99 211	80	0.12 533	623	0.230
0.021	0.99 131	85	0.13 156	623	0.229
0.022	0.99 046	88	0.13 779	622	0.228
0.023	0.98 958	93	0.14 401	622	0.227
0.024	0.98 865	96	0.15 023	620	0.226
0.025	0.98 769	100	0.15 643	621	0.225
0.026	0.98 669	105	0.16 264	619	0.224
0.027	0.98 564	108	0.16 883	619	0.223
0.028	0.98 456	111	0.17 502	619	0.222
0.029	0.98 345	116	0.18 121	617	0.221
0.030	0.98 229	120	0.18 738	617	0.220
0.031	0.98 109	123	0.19 355	616	0.219
0.032	0.97 986	128	0.19 971	615	0.218
0.033	0.97 858	131	0.20 586	615	0.217
0.034	0.97 727	135	0.21 201	613	0.216
0.035	0.97 592	139	0.21 814	613	0.215
0.036	0.97 453	143	0.22 427	612	0.214
0.037	0.97 310	147	0.23 039	611	0.213
0.038	0.97 163	150	0.23 650	610	0.212
0.039	0.97 013	155	0.24 260	609	0.211
0.040	0.96 858	158	0.24 869	608	0.210
0.041	0.96 700	162	0.25 477	607	0.209
0.042	0.96 538	166	0.26 084	606	0.208
0.043	0.96 372	169	0.26 690	605	0.207
0.044	0.96 203	174	0.27 295	604	0.206
0.045	0.96 029	177	0.27 899	603	0.205
0.046	0.95 852	181	0.28 502	602	0.204
0.047	0.95 671	185	0.29 104	600	0.203
0.048	0.95 486	188	0.29 704	600	0.202
0.049	0.95 298	192	0.30 304	598	0.201
0.050	0.95 106	196	0.30 902	597	0.200
0.051	0.94 910	200	0.31 499	595	0.199
0.052	0.94 710	204	0.32 094	595	0.198
0.053	0.94 506	207	0.32 689	593	0.197
0.054	0.94 299	211	0.33 282	592	0.196
0.055	0.94 088	215	0.33 874	590	0.195
0.056	0.93 873	218	0.34 464	589	0.194
0.057	0.93 655	222	0.35 053	588	0.193
0.058	0.93 433	226	0.35 641	587	0.192
0.059	0.93 207	229	0.36 228	584	0.191
0.060	0.92 978	233	0.36 812	584	0.190
0.061	0.92 745	237	0.37 396	582	0.189
0.062	0.92 508	241	0.37 978	580	0.188
0.063	0.92 267		0.38 558		0.187
--	sin E		cos E		E

E	cos E		sin E		--
0.062	0.92 508	241	0.37 978	580	0.188
0.063	0.92 267	244	0.38 558	579	0.187
0.064	0.92 023	248	0.39 137	578	0.186
0.065	0.91 775	251	0.39 715	576	0.185
0.066	0.91 524	255	0.40 291	574	0.184
0.067	0.91 269	258	0.40 865	573	0.183
0.068	0.91 011	263	0.41 438	571	0.182
0.069	0.90 748	265	0.42 009	569	0.181
0.070	0.90 483	270	0.42 578	568	0.180
0.071	0.90 213	272	0.43 146	566	0.179
0.072	0.89 941	277	0.43 712	564	0.178
0.073	0.89 664	280	0.44 276	562	0.177
0.074	0.89 384	283	0.44 838	561	0.176
0.075	0.89 101	287	0.45 399	559	0.175
0.076	0.88 814	291	0.45 958	557	0.174
0.077	0.88 523	294	0.46 515	555	0.173
0.078	0.88 229	297	0.47 070	554	0.172
0.079	0.87 932	301	0.47 624	551	0.171
0.080	0.87 631	305	0.48 175	550	0.170
0.081	0.87 326	308	0.48 725	548	0.169
0.082	0.87 018	311	0.49 273	546	0.168
0.083	0.86 707	315	0.49 819	543	0.167
0.084	0.86 392	318	0.50 362	542	0.166
0.085	0.86 074	321	0.50 904	540	0.165
0.086	0.85 753	325	0.51 444	538	0.164
0.087	0.85 428	329	0.51 982	535	0.163
0.088	0.85 099	331	0.52 517	534	0.162
0.089	0.84 768	335	0.53 051	532	0.161
0.090	0.84 433	339	0.53 583	529	0.160
0.091	0.84 094	341	0.54 112	527	0.159
0.092	0.83 753	345	0.54 639	526	0.158
0.093	0.83 408	348	0.55 165	523	0.157
0.094	0.83 060	352	0.55 688	520	0.156
0.095	0.82 708	355	0.56 208	519	0.155
0.096	0.82 353	358	0.56 727	516	0.154
0.097	0.81 995	361	0.57 243	514	0.153
0.098	0.81 634	365	0.57 757	512	0.152
0.099	0.81 269	367	0.58 269	510	0.151
0.100	0.80 902	371	0.58 779	507	0.150
0.101	0.80 531	374	0.59 286	505	0.149
0.102	0.80 157	378	0.59 791	502	0.148
0.103	0.79 779	380	0.60 293	500	0.147
0.104	0.79 399	383	0.60 793	498	0.146
0.105	0.79 016	387	0.61 291	495	0.145
0.106	0.78 629	390	0.61 786	493	0.144
0.107	0.78 239	393	0.62 279	490	0.143
0.108	0.77 846	396	0.62 769	488	0.142
0.109	0.77 450	399	0.63 257	485	0.141
0.110	0.77 051	402	0.63 742	483	0.140
0.111	0.76 649	405	0.64 225	481	0.139
0.112	0.76 244	408	0.64 706	477	0.138
0.113	0.75 836	411	0.65 183	476	0.137
0.114	0.75 425	414	0.65 659	472	0.136
0.115	0.75 011	417	0.66 131	470	0.135
0.116	0.74 594	420	0.66 601	468	0.134
0.117	0.74 174	423	0.67 069	464	0.133
0.118	0.73 751	425	0.67 533	462	0.132
0.119	0.73 326	429	0.67 995	460	0.131
0.120	0.72 897	432	0.68 455	456	0.130
0.121	0.72 465	434	0.68 911	454	0.129
0.122	0.72 031	437	0.69 365	452	0.128
0.123	0.71 594	440	0.69 817	448	0.127
0.124	0.71 154	443	0.70 265	446	0.126
0.125	0.70 711		0.70 711		0.125
--	sin E		cos E		E

Sheet 2: $i - jn'/n = i - 0.44271096j$

i \ j	j = 0	1	2	3	4	5
-3		-3.44271096	-3.88542192			
-2		-2.44271096	-2.88542192	-3.328133	-3.770844	
-1		-1.44271096 (1,1)	-1.88542192	-2.328133	-2.770844	
0	0.0 (3,3)	-0.44271096 (2,2)	-0.88542192 (1,1)	-1.328133 (1,1)	-1.770844 (1,0)	-2.213555
1	+1.0 (3,3)	+0.55728904 (2,2)	+0.11457808 (2,1)	-0.328133 (1,1)	-0.770844 (1,1)	-1.213555 (1,1)
2	+2.0 (2,2)	+1.55728904 (1,1)	+1.11457808 (1,1)	+0.671867 (1,1)	+0.229156 (2,1)	-0.213555 (2,1)
3	+3.0 (1,1)	+2.55728904	+2.11457808	+1.671867 (1,1)	+1.229156 (1,1)	+0.786445 (1,1)
4	+4.0	+3.55728904	+3.11457808	+2.671867	+2.229156	+1.786445 (1,1)
5	+5.0	+4.55728904	+4.11457808	+3.671867	+3.229156	+2.786445
6		+5.55728904	+5.11457808	+4.671867	+4.229156	+3.786445
7				+5.671867	+5.229156	+4.786445
8						+5.786445
9						
10						
11						
12						
	(3,3)	(2,2)	(2,1)	(1,1)	(2,1)	(2,1)

Sheet 3:

1	0	30	60	90	120	150
2	2.7372494	2.7752924	2.8792277	3.0212060	3.1631843	3.2671197
3	+13.433798	+26.290805	+32.502400	+30.404190	+20.558391	+5.603175
4	-25.850543	-16.053831	-1.236954	+14.629918	+27.295271	+33.365432
5	35.083867	35.913980	36.801417	37.537905	37.940856	37.887547
6	+13.398943	+26.228074	+32.430981	+30.340951	+20.516378	+5.591537
7	-25.917964	-16.092320	-1.239684	+14.660475	+27.351281	+33.435023
8	35.133603	35.931218	36.801509	37.549835	37.981190	37.948861
9	851.272531	946.874629	1053.305345	1135.502835	1169.014339	1149.166049
10	29.176575	30.771328	32.454666	33.697223	34.190852	33.899352
11	19.572877	18.552029	17.350669	16.567658	16.539542	17.057257
12	1.2916547	1.2683914	1.2393162	1.2192611	1.2185235	1.2319092
13	0.5333294	0.5647851	0.5993234	0.6226678	0.6271165	0.6162833
14	0.3975073	0.4683904	0.5605214	0.6332269	0.6481934	0.6123964
15	9.7269955	9.7518832	9.7776612	9.7942564	9.7973482	9.7897804
16	7.29138035	7.3030120	7.3175496	7.32757715	7.3279460	7.3212531
17	6.52260435	6.5574993	6.6011121	6.63119475	6.6323012	6.6122226
18	-0.5169751	-0.6135532	-0.0382253	+0.4831910	+0.7501067	+0.1672359
19	297.33784	298.46867	297.81092	295.78941	293.12619	290.50590
20	0.00324038	0.00307244	0.00291308	0.00280566	0.00276516	0.00278893
21	+0.00148810	+0.00146457	+0.00135911	+0.00122064	+0.00108604	+0.00097698
22	-0.00287848	-0.00270091	-0.00257660	-0.00252621	-0.00254296	-0.00261221
23	437	393	354	328	319	324
	+583,535	+72,111	+7,619	+1,055	+20,417	-1,739
	+583,534	+72,116	+7,612	-1,176	+20,422	-1,610
	-1403,060	-20,417	+1,739	+1,177	+72,112	+7,619
	-1403,056	-20,422	+1,610	+1,054	+72,116	+7,612
	+1167,069	+144,227	+15,231	+1,055	+7	-5
	(+1)	+40,839	-3,349	-1,176	-129	-5
	-2806,116	-40,839	+3,349	+1,176	+129	+5
	(-4)	+144,228	+15,231	+1,054	+7	-4

See page 136 for the designation of the quantities on each line.

Sheet 2 (cont.):

i	6	7	8	9	11	12
0						
1	-1.656266 (1,1)	-2.098977				
2	-0.656266 (1,1)	-1.098977 (1,1)	-1.54169 (1,1)	-1.98440 (1,1)		
3	+0.343734 (1,1)	-0.098977 (2,1)	-0.54169 (1,1)	-0.98440 (2,2)	-1.42711 (1,1)	-1.86982 (1,1)
4	+1.343734 (1,1)	+0.901023 (1,1)	+0.45831 (2,1)	+0.01560 (3,2)	-0.42711 (2,1)	-0.86982 (1,1)
5	+2.343734	+1.901023 (1,1)	+1.45831 (1,1)	+1.01560 (2,2)	+0.57289 (1,1)	+0.13018 (2,1)
6	+3.343734	+2.901023	+2.45831	+2.01560 (1,1)	+1.57289 (1,1)	+1.13018 (1,1)
7	+4.343734	+3.901023	+3.45831	+3.01560	+2.57289	+2.13018
8	+5.343734	+4.901023	+4.45831	+4.01560	+3.57289	+3.13018
9		+5.901023	+5.45831	+5.01560	+4.57289	+4.13018
10			+6.45831	+6.01560	+5.57289	+5.13018
11				+7.01560	+6.57289	+6.13018
12					+7.57289	+7.13018
	(1,1)	(2,1)	(2,1)	(3,2)	(2,1)	(2,1)

Sheet 3 (cont.):

180	210	240	270	300	330	1
3.3051627	3.2671197	3.1631843	3.0212060	2.8792277	2.7752924	2
-10.454220	-23.311226	-29.522821	-27.424612	-17.578812	-2.623596	3
+31.213907	+21.417196	+6.600319	-9.266554	-21.931906	-28.002067	4
37.362746	36.492317	35.524248	34.747445	34.384809	34.518750	5
-10.431594	-23.257782	-29.450440	-27.352813	-17.530725	-2.616425	6
+31.281757	+21.466524	+6.616581	-9.290942	-21.992230	-28.079023	7
37.419467	36.521311	35.527277	34.753965	34.423349	34.581236	8
1087.366474	1001.736076	911.107560	834.497982	790.984499	795.277212	9
32.975240	31.650214	30.184558	28.887679	28.124447	28.200660	10
17.687567	18.222790	18.737125	19.322011	19.848991	20.014622	11
1.2476681	1.2606149	1.2727030	1.2860523	1.2977384	1.3013474	12
0.5983853	0.5781484	0.5562497	0.5342054	0.5182096	0.5165348	13
0.5577900	0.5020779	0.4480450	0.3993361	0.3671310	0.3638996	14
9.7769809	9.7620393	9.7452698	9.7277083	9.7145055	9.7130996	15
7.31337365	7.30690025	7.3008562	7.29418155	7.2883385	7.2865340	16
6.58858425	6.56916405	6.5510319	6.53100795	6.5134788	6.5080653	17
-0.3334721	-0.9229824	-0.2246683	+0.3396704	+0.7971327	+0.0931808	18
288.44210	287.29354	287.33771	288.76110	291.44050	294.67651	19
0.00286709	0.00298712	0.00313217	0.00327278	0.00336160	0.00335251	
+0.00090699	+0.00088797	+0.00093340	+0.00105260	+0.00122878	+0.00139965	
-0.00271985	-0.00285209	-0.00298986	-0.00309889	-0.00312897	-0.00304636	
342	372	409	446	471	468	

$$[1 - q_1\cos(Q + E')]^{-3/2} = +1.00000389 + \Sigma\ (C,k,l)\cos(kE - lE') + (S,k,l)\sin(kE - lE')$$

k	l	(C,k,l)	(S,k,l)
-1	1	+1167,1	-2806,1
-2	1	+288,5	- 81,7
-3	1	+30,5	+ 6,7
-4	1	+2,1	+2,4

j	E = 0	30	60	90	120	150
Sheet 4:			log cos j Q'			
1	9.662037	9.678225	9.668903	9.638554	9.594124	9.544446
2	9.762081n	9.736839n	9.751781n	9.793398n	9.839782n	9.877702n
3	9.99577n	9.99860n	9.99714n	9.98936n	9.97124n	9.94397n
			log sin j Q'			
1	9.948567n	9.944027n	9.946694n	9.954435n	9.963618n	9.971571n
2	9.911633n	9.923283n	9.916627n	9.894019n	9.858774n	9.817047n
3	9.14283	8.90360	9.05831	9.33982	9.54678	9.67841
Sheet 5:			log $b_{1/2}^{(j)}$ cos j Q'			
0	7.5553123	7.5613311	7.5689200	7.5737151	7.5730324	7.5688688
1	6.661089	6.710742	6.737906	6.731263	6.689705	6.627182
2	6.369315n	6.403339n	6.481672n	6.564434n	6.616950n	6.634051n
3	6.25396n	6.34143n	6.42974n	6.48015n	6.47135n	6.41547n
			log $b_{1/2}^{(j)}$ sin j Q'			
1	6.947619n	6.976543n	7.015697n	7.047144n	7.059199n	7.054307n
2	6.518867n	6.589783n	6.646518n	6.665055n	6.635942n	6.573396n
3	5.40102	5.24643	5.49091	5.83061	6.04689	6.14991
Sheet 6:			log $b_{3/2}^{(j)}$ cos j Q'			
0	6.9266989	6.9759690	7.0370096	7.0799623	7.0836299	7.0573766
1	6.445085	6.529675	6.600441	6.624843	6.586235	6.505006
2	6.351698n	6.417336n	6.535240n	6.646073n	6.700949n	6.700768r
3	6.37021n	6.48747n	6.61310n	6.68991n	6.68313n	6.61078n
			log $b_{3/2}^{(j)}$ sin j Q'			
1	6.731615n	6.795477n	6.878232n	6.940724n	6.955729n	6.932131n
2	6.501250n	6.603780n	6.700086n	6.746694n	6.719941n	6.640113n
3	5.51727	5.39247	5.67427	6.04037	6.25867	6.34522
Sheet 7:			$b_{1/2}^{(j)}$ cos j Q'			
0	+3591,802	+3641,926	+3706,125	+3747,271	+3741,385	+3705,687
1	+458,24	+513,74	+546,90	+538,60	+489,45	+423,82
2	-234,05	-253,13	-303,16	-366,80	-413,95	-430,58
3	-179,5	-219,5	-269,0	-302,1	-296,0	-260,3
			$b_{1/2}^{(j)}$ sin j Q'			
1	-886,38	-947,42	-1036,80	-1114,66	-1146,04	-1133,20
2	-330,27	-388,85	-443,12	-462,44	-432,46	-374,45
3	+25,2	+17,6	+31,0	+67,7	+111,4	+141,2
Sheet 8:			$b_{3/2}^{(j)}$ cos j Q'			
0	+844,693	+946,170	+1088,954	+1202,160	+1212,355	+1141,239
1	+278,67	+338,59	+398,51	+421,54	+385,69	+319,89
2	-224,7	-261,4	-343,0	-442,7	-502,3	-502,1
3	-234,5	-307,2	-410,3	-489,7	-482,1	-408,1
			$b_{3/2}^{(j)}$ sin j Q'			
1	-539,03	-624,42	-755,50	-872,42	-903,09	-855,32
2	-317,1	-401,6	-501,3	-558,1	-524,7	-436,6
3	+32,9	+24,7	+47,2	+109,7	+181,4	+221,4

These Sheets are incomplete; they should be carried to j = 11.

180	210	240	270	300	330
9.500162	9.473147	9.474221	9.507347	9.562928	9.620651
9.903009n	9.915540n	9.915077n	9.899338n	9.864967n	9,813846n
9.91509n	9.89582n	9.89661n	9.92001n	9.95484n	9.98291n
9.977103n	9.979910n	9.979806n	9.976290n	9.968855n	9.958410n
9.778296n	9.754087n	9.755057n	9.784666n	9.832814n	9.880091n
9.75504	9.79050	9.78921	9.74436	9.63681	9.43956
7.5649448	7.5626351	7.5607614	7.5579655	7.5547528	7.5532152
6.564420	6.518290	6.498835	6.509833	6.547788	6.602445
6.627442n	6.605242n	6.566803n	6.510743n	6.445114n	6.389477n
6.34153n	6.27223n	6.21792n	6.18310n	6.17324n	6.19537n
7.041361n	7.025053n	7.004420n	6.978776n	6.953715n	6.940204n
6.502729n	6.443789n	6.406783n	6.396071n	6.412961n	6.455722n
6.18148	6.16691	6.11052	6.00745	5.85521	5.65202
7.0239837	6.9941615	6.9654670	6.9354849	6.9111261	6.9050160
6.418182	6.350393	6.310152	6.299738	6.320555	6.371050
6.672353n	6.630899n	6.574117n	6.498947n	6.417756n	6.358118n
6.51630n	6.42909n	6.35778n	6.30512n	6.28056n	6.29877n
6.895123n	6.857156n	6.815737n	6.768681n	6.726482n	6.708809n
6.547640n	6.469446n	6.414097n	6.384275n	6.385603n	6.424363n
6.35625	6.32377	6.25038	6.12947	5.96253	5.75542
+3672,356	+3652,877	+3637,152	+3613,812	+3587,177	+3574,499
+366,79	+329,83	+315,38	+323,47	+353,01	+400,35
-424,07	-402,94	-368,81	-324,15	-278,69	-245,18
-219,5	-187,2	-165,2	-152,4	-149,0	-156,8
-1099,92	-1059,38	-1010,23	-952,30	-898,91	-871,37
-318,22	-277,84	-255,14	-248,93	-258,80	-285,58
+151,9	+146,9	+129,0	+101,7	+71,6	+44,9
+1056,778	+986,646	+923,564	+861,956	+814,941	+803,556
+261,93	+224,07	+204,25	+199,41	+209,20	+234,99
-470,3	-427,5	-375,1	-315,5	-261,7	-228,1
-328,3	-268,6	-227,9	-201,9	-190,8	-199,0
-785,46	-719,71	-654,24	-587,06	-532,70	-511,46
-352,9	-294,7	-259,5	-242,3	-243,0	-265,7
+227,1	+210,8	+178,0	+134,7	+91,7	+56,9

The numbered lines of Sheet 3 on pages 132 and 133 contain the following quantities:

1) E	6) q cosQ	11) $\sqrt{C^2 - q^2}$	16) (P,1/2)	20) 1.5 w/q
2) r	7) q sin Q	12) $\log\sqrt{C^2 - q^2}$	17) (P,3/2)	21) 1.5 w/q cosQ'
3) K cos ψ	8) C	13) A	18) tan or cot Q	22) 1.5 w/q sin Q'
4) K sin ψ	9) q^2	14) p	19) Q'	23) $0.003724/q^2$
5) H	10) q	15) log A		

Sheet 9: Harmonic analysis

j	$(C_0 + C_6)/2$ $(C_0 - C_6)/2$	$(C_1 + C_5)/4$ $(C_1 - C_5)/4$	$(C_2 + C_4)/4$ $(C_2 - C_4)/4$	$C_3/2$ $S_3/2$	$(S_1 + S_5)/4$ $(S_1 - S_5)/4$	$(S_2 + S_4)/4$ $(S_2 - S_4)/4$
0	+1827,9998 +1828,0060	-10,2643 -10,2580	-5,9801 -6,1324	+0.3901 -1,1018	+16,1315 +16,1067	+1,0549 +1,0620
1	+210,814 +210,818	+11,582 +11,579	-2,278 -2,350	-0,302 -0,646	+26,568 +26,555	+1,400 +1,430
2	-168,561 -168,565	+24,206 +24,192	+2,015 +2,086	-0,908 +0,588	-5,037 -5,024	+1,421 +1,492
3	-106,52 -106,52	+5,13 +5,14	+3,38 +3,55	-0,27 +1,16	-18,13 -18,10	+0,75 +0,78
1	-506,523 -506,528	+26,985 +26,976	+4,974 +5,106	-0,585 +1,041	-19,775 -19,753	-0,161 -0,150
2	-169,834 -169,841	-1,601 -1,598	+3,856 +4,001	+0,189 +1,136	-26,121 -26,099	-0,481 -0,505
3	+43,34 +43,33	-16,30 -16,28	+0,47 +0,49	+0,92 +0,08	-4,21 -4,20	-1,56 -1,66
0	+495,1071 +495,1439	-27,3414 -27,2913	-9,8697 -10,4425	+1,6616 -3,5831	+40,7339 +40,6169	-0,8645 -1,0665
1	+144,854 +144,874	+2,135 +2,138	-4,852 -5,182	-0,086 -1,892	+26,820 +26,757	+0,561 +0,568
2	-181,42 -181,44	+31,83 +31,76	+3,84 +4,05	-2,26 +1,61	-15,10 -15,05	+2,98 +3,31
3	-156,16 -156,21	+12,35 +12,30	+7,73 +8,35	-1,26 +3,34	-34,30 -34,19	+2,26 +2,50
1	-347,502 -347,532	+31,750 +31,693	+8,190 +8,669	-1,892 +3,066	-34,137 -34,038	+1,635 +1,880
2	-183,21 -183,25	+4,65 +4,62	+7,85 +8,42	-0,34 +3,17	-37,89 -37,78	+0,43 +0,50
3	+63,19 +63,18	-25,37 -25,30	+0,90 +1,04	+2,19 +0,28	-2,98 -2,97	-3,09 -3,46

Sheet 10:

C_0	$\frac{1}{2}C_1$	$\frac{1}{2}C_2$	$\frac{1}{2}C_3$	$\frac{1}{2}C_4$	$\frac{1}{2}C_5$
(C_6)	$\frac{1}{2}S_1$	$\frac{1}{2}S_2$	$\frac{1}{2}S_3$	$\frac{1}{2}S_4$	$\frac{1}{2}S_5$
C_0'	$\frac{1}{2}C_1'$	$\frac{1}{2}C_2'$	$\frac{1}{2}C_3'$	$\frac{1}{2}C_4'$	$\frac{1}{2}C_5'$
(C_6')	$\frac{1}{2}S_1'$	$\frac{1}{2}S_2'$	$\frac{1}{2}S_3'$	$\frac{1}{2}S_4'$	$\frac{1}{2}S_5'$

0	+3656,0058	-20,5223	-12,1125	+0,3901	+0,1523	-0,0063
	(-62)	+32,2382	+2,1169	-1,1018	-0,0071	+0,0248
	0	0	0	0	0	0
	0	0	0	0	0	0
1	+421,632	+23,161	-4,628	-0,302	+0,072	+0,003
	(-4)	+53,123	+2,830	-0,646	-0,030	+0,013
	-1013,051	+53,961	+10,080	-0,585	-0,132	+0,009
	(+5)	-39,528	-0,311	+1,041	-0,011	-0,022
2	-337,126	+48,398	+4,101	-0,908	-0,071	+0,014
	(+4)	-10,061	+2,913	+0,588	-0,071	-0,013
	-339,675	-3,199	+7,857	+0,189	-0,145	-0,003
	(+7)	-52,220	-0,986	+1,136	+0,024	-0,022
3	-213,04	+10,27	+6,93	-0,27	-0,17	-0,01
	(0)	-36,23	+1,53	+1,16	-0,03	-0,03
	+86,67	-32,58	+0,96	+0,92	-0,02	-0,02
	(+1)	-8,41	-3,22	+0,08	+0,10	-0,01

0	+990,2510	-54,6327	-20,3122	+1,6616	+0,5728	-0,0501
	(-368)	+81,3508	-1,9310	-3,5831	+0,2020	+0,1170
	0	0	0	0	0	0
	0	0	0	0	0	0
1	+289,728	+4,273	-10,034	-0,086	+0,330	-0,003
	(-20)	+53,577	+1,128	-1,892	-0,006	+0,063
	-695,034	+63,443	+16,859	-1,892	-0,479	+0,057
	(+30)	-68,175	+3,515	+3,066	-0,245	-0,099
2	-362,86	+63,59	+7,89	-2,26	-0,21	+0,07
	(+2)	-30,15	+6,29	+1,61	-0,33	-0,05
	-366,46	+9,27	+16,27	-0,34	-0,57	+0,03
	(+4)	-75,67	+0,93	+3,17	-0,07	-0,11
3	-312,37	+24,65	+16,08	-1,26	-0,62	+0,05
	(+5)	-68,49	+4,76	+3,34	-0,24	-0,11
	+126,37	-50,67	+1,94	+2,19	-0,14	-0,07
	(+1)	-5,95	-6,55	+0,28	+0,37	-0,01

Sheet 11: $3m'a[C - q\cos(Q - E')]^{-1/2} : (iE - jE')$

j	0		1		2		3	
i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-1,	+1,	0	0		
-1			-4,3	-12,9	-2,	-1,	0	0
0	+3656,006/2	0,0	+62,69	-107,08	+5,1	-10,8	-0,4	-2,1
1	-20,522	+32,238	+421,63	+1013,05	+100,62	+13,26	+10,2	-2,5
2	-12,11	+2,12	-16,4	-0,8	-337,1	+339,7	+18,7	+68,8
3	+0,4	-1,1	-5,	-7,	-4,	-7,	-213,0	-86,7
4	0,	0,	+1,	0,	+3,	-5,	+2,	-4,
5			0,	0,	0,	0,	+4,	+1,

$10m'a[C - q\cos(Q - E')]^{-3/2} : (iE - jE')$

-2			-3,	+4,	0,	+1,		
-1			-13,5	-18,0	-5,	-1,	-1,	0,
0	+990,251/2	0,0	+72,45	-117,02	+7,0	-22,6	-1,5	-5,5
1	-54,633	+81,351	+289,73	+695,03	+139,3	+20,9	+22,6	-6,7
2	-20,31	-1,93	-63,9	-9,9	-362,9	+366,5	+30,6	+119,2
3	+1,7	-3,6	-7,	-16,	-12,	-39,	-312,4	-126,4
4	+1,	0,	+3,	0,	+9,	-10,	+19,	-18,
5			0,	0,	+1,	+2,	+10,	+3,

Sheet 12: $3m'a\Omega$:(iE - jE')

j	0		1		2		3	
i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-1,	+1,				
-1			-3,5	-8,7	-2,	-1,	0,	0,
0	+3657,689/2	0,0	+97,31	-24,39	+8,2	-4,8	-0,3	-1,8
1	-29,345	+53,755	+53,33	+127,50	+73,96	-50,83	+9,0	-5,4
2	-12,50	+2,51	-16,4	-0,8	-337,1	+339,7	+18,7	+68,8
3	+0,4	-1,1	-5,	-7,	-4,	-7,	-213,0	-86,7
4			+1,	0,	-3,	-5,	+2,	-4,
5					0,	0,	+4,	+1,

$10m'a\Delta^{-3}$:(iE - jE')

-2			-3,	+4,	0,	+1,	0,	0,
-1			-12,9	-19,4	-5,	-1,	-1,	0,
0	+990,288/2	0,0	+72,53	-116,90	+8,1	-22,6	-1,4	-5,6
1	-54,435	+81,383	+289,75	+695,06	+139,3	+21,0	+22,9	-6,0
2	-21,10	-1,13	-63,9	-9,7	-362,9	+366,5	+30,5	+119,2
3	+1,7	-3,6	-8,	-16,	-12,	-39,	-312,4	-126,4
4	+1,	0,	+3,	0,	+9,	-10,	+19,	-18,
5			0,	0,	+1,	+2,	+10,	+3,
(0	+577,772/2	0,0	+42,71	-116,90	+6,7	-22,6	-1,5	-5,6)

(-3H)

-1			+0,0656	+5,8904	+0,015	+0,400		
0	+1,6702/2	0,0	+34,6103	+82,6784	+2,504	+5,987	+0,12	+0,29
1	-8,8850	+21,5091	-368,3081	-885,5617	-26,659	-64,099	-1,29	-3,09

Sheet 13: $\partial(3a\Omega)/\partial E$:(iE - jE')

-2			-2,	-2,	0,	0,		
-1			+8,7	-3,5	+1,	-2,	0,	0,
0	0,000	0,000	0,00	0,00	0,0	0,0	0,0	0,0
1	+53,755	+29,345	+127,50	-53,33	-50,83	-73,96	-5,4	-9,0
2	+5,02	+25,00	-1,5	+32,8	+679,4	+674,2	+137,6	-37,4
3	-3,3	-1,2	-21,	+15,	-21,	+12,	-260,1	+639,0
4	0,	0,	0,	-4,	-20,	+12,	-16,	-8,
5					0,	0,	+5,	-20,

$a\,r\ \partial\Omega/\partial r$:(iE - jE')

-2			-2,	+3,	0,	+1,		
-1			-4,2	-13,7	-3,	0,	-1,	0,
0	+265,057/2	0,0	+12,91	-28,86	+9,4	-8,0	-0,6	-3,5
1	-24,709	+28,160	+61,62	+147,11	+63,5	-34,9	+10,8	-1,6
2	-16,01	+3,09	-17,5	-3,1	-261,6	+264,1	+34,5	+64,8
3	+1,1	-2,2	-9,	-12,	-5,	-9,	-236,6	-95,9
4	+1,	0,	+2,	0,	+7,	-10,	+3,	-6,
5					+1,	+1,	+8,	+1,

Sheet 16: $a^2\ \partial\Omega/\partial Z$:(iE - jE')

-2			+1,	+3,	+1,	0,		
-1			-8,5	+9,5	-2,	+2,	0,	+1,
0	+29,779/2	0,0	-18,35	-74,99	-16,2	-1,4	-3,1	-0,2
1	-99,581	+13,985	+5,88	+24,28	+81,6	-60,5	-1,9	-18,5
2	+3,14	-7,54	-34,7	-58,6	-17,1	+9,7	+59,2	+36,4
3	+2,4	-0,6	+5,	0,	+26,	-35,	-10,8	-9,9
4	0,	0,	+1,	+1,	+2,	+3,	+27,	+7,
5					-1,	+1,	-1,	+2,

Sheet 14:

$\partial(3a\Omega)/\partial E$:(iE - jg')

j	0		1		2		3	
i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-2,	-2,	0,	0,		
-1			+8,6	-3,4	+1,	-2,	0,	0,
0	0,000	0,000	0,00	0,00	0,0	0,0	0,0	0,0
1	+50,469	+30,547	+129,87	-49,74	-47,25	-74,43	-7,6	-12,4
2	+5,10	+24,16	-34,2	+0,2	+667,9	+676,1	+168,4	-3,5
3	-2,8	-1,6	-20,	+15,	-2,	-34,	-268,0	+625,5
4	0,	0,	+1,	-5,	-20,	+12,	+28,	-7,
5			0,	0,	0,	+1,	+4,	-20,

$a r\ \partial\Omega/\partial r$:(iE - jg')

i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-2,	+2,	0,	+1,		
-1			-4,0	-13,6	-3,	-1,	-1,	0,
0	+264,434/2	0,0	+12,45	-28,47	+9,7	-8,4	-0,1	-3,8
1	-26,096	+24,280	+58,53	+148,70	+64,1	-31,2	+13,6	-2,9
2	-15,54	+3,10	-4,8	-15,8	-263,9	+258,7	+21,0	+76,4
3	+1,3	-1,9	-9,	-12,	+12,	-2,	-231,6	-100,3
4	+1,	0,	+1,	0,	+7,	-9,	+3,	+9,
5					0,	+1,	+8,	+1,

Sheet 15:

$\partial(3a\Omega)/\partial E$:(iE - jϕ)

i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-2,	-2,	0,	0,		
-1			+8,6	-3,5	+1,	-2,	0,	0,
0	0,000	0,0	-2,53	+0,96	+2,6	+3,6	+0,8	+0,7
1	+50,469	+30,547	+130,52	-49,72	-74,93	-102,43	-18,6	-10,9
2	+5,10	+24,16	-31,0	-1,1	+664,8	+673,2	+184,0	-43,3
3	-2,8	-1,6	-21,	+15,	+26,	-6,	-258,2	+623,2
4	0,	0,	+1,	-4,	-19,	+12,	+11,	+33,
5			0,	0,	-1,	+2,	+6,	-19,

$a r\ \partial\Omega/\partial r$:(iE - jϕ)

i	cos	sin	cos	sin	cos	sin	cos	sin
-2			-2,	+3,	0,	+1,		
-1			-4,3	-12,9	-3,	-1,	0,	0,
0	+264,434/2	0,0	+11,14	-31,84	+6,7	-6,9	-1,0	-3,5
1	-26,096	+24,280	+58,86	+148,37	+75,4	-42,2	+11,8	-8,1
2	-15,54	+3,10	-3,4	-12,5	-261,3	+257,1	+36,1	+82,2
3	+1,3	-1,9	-9,	-12,	+1,	+9,	-229,5	-95,7
4	+1,	0,	+1,	0,	+7,	-9,	-12,	+3,
5					+1,	+1,	+8,	+2,

Sheet 17:

$a^2\ \partial\Omega/\partial Z$:(iE - jg')

i	cos	sin	cos	sin	cos	sin	cos	sin
-2			+1,	+3,	+1,	0,		
-1			-8,4	+9,4	-2,	+2,	0,	+1,
0	+30,665/2	0,0	-17,56	-74,90	-16,4	-3,2	-3,8	-0,4
1	-99,518	+13,629	+1,94	+27,17	+81,6	-58,4	+2,2	-21,0
2	+3,95	-6,05	-33,8	-59,0	-22,1	+5,7	+56,7	+37,4
3	+2,3	-0,6	+4,	+2,	+27,	-34,	-8,7	-15,8
4	0,	0,	+1,	+1,	0,	+3,	+26,	+8,

$a^2\partial\Omega/\partial Z$:(iE - jϕ)

i	cos	sin	cos	sin	cos	sin	cos	sin
-2			+1,	+3,	+1,	0,		
-1			-8,0	+11,0	-1,	+2,	0,	+1,
0	+30,665/2	0,0	-17,77	-75,24	-19,9	-0,7	-3,8	+1,0
1	-99,518	+13,629	+2,28	+26,81	+81,8	-58,7	-1,6	-23,3
2	+3,95	-6,05	-33,8	-58,5	-19,9	+4,6	+57,2	+36,9
3	+2,3	-0,6	+3,	+1,	+26,	-34,	-6,8	-13,9
4	0,	0,	+1,	+2,	+1,	+1,	+26,	+7,

Sheet 18:

		cos		W		sin			
i	j	h = - 1	0	+ 1	- 1	0	+ 1	i	j
0	0							0	0
0		+58,214 E/2	+5,471 E/2	+58,214 E/2	-32,895 E/2	0,000	+32,895 E/2	0	
1		-264,752	+16,560	-32,877	-5,471	-50,471	+5,438	1	
1								1	
2		+10,07	+12,62	+1,39	+4,54	-2,43	-1,96	2	
2								2	
3		-0,5	-0,5	+0,3	+0,3	+0,9	0,0	3	
4	0	-0,2	0,		0,	0,		4	0
		-0,2		-0,2					
-3	1		0,	+0,1		0,	-0,6	-3	1
-2		+0,4	+0,9	+1,1	0,	-0,7	+2,2	-2	
-1		-2,4	+2,1	-4,6	+3,0	+7,5	+30,4	-1	
0		-14,4	-12,8	-210,6	-48,3	-33,1	-533,9	0	
1		-31,0	-90,8	-2,9	+29,1	-235,5	-56,5	1	
2		-16,2	-2,1	-12,6	-39,1	+17,3	-16,5	2	
3		+2,9	+6,1	+1,6	-0,7	+8,2	+0,3	3	
4	1	-0,4	-1,1	0,	-0,6	-0,2		4	1
		+0,3					-0,1		
-2	2		0,	+0,8		0,	-0,2	-2	2
-1		-0,1	+1,1	+0,6	+0,3	+0,6	+1,1	-1	
0		-5,9	-13,3	-189,8	-1,2	+9,0	+130,8	0	
1		-174,0	-1195,7	-6246,5	+142,5	+949,9	+6154,4	1	
2		+33,8	+606,7	+21,7	-46,6	-598,3	+7,2	2	
3		-88,8	-7,0	-0,4	+87,7	-8,7	-11,2	3	
4		+0,5	+3,7	+0,2	+3,7	+6,4	+0,1	4	
5		-3,6	+0,3		-0,9	+0,2		5	
6	2	-0,4			-0,3			6	2
				+0,1					
-1	3		0,	+1,0			+1,0	-1	3
0		-0,5	-1,2	-12,6	0,	+1,6	+20,8	0	
1		+1,1	+20,9	-262,5	-18,0	-88,3	-655,8	1	
2		-15,3	-110,5	-964,6	-26,7	-294,3	-408,7	2	
3		+20,0	+373,2	-9,8	+34,5	+156,6	+10,7	3	
4		-69,1	+9,5	+8,2	-28,7	-5,4	+2,1	4	
5		-3,4	-5,3	0,	+1,0	-1,6	-0,2	5	
6	3	+1,0	-0,2		+0,4	+0,2	0,	6	3
		+0,1			-0,1				
. . .	. . .	. . . .	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .	. .	.
3	9		-0,1	-0,6		-0,1	-0,4	3	9
4		-5,9	-40,8	-111,2	-1,7	-13,6	+71,1	4	
5		+0,3	+1,3	-2,7	-0,1	-1,1	-7,2	5	
6		0,0	+1,6	+11,9	+0,3	+3,2	+0,7	6	
7		-0,7	-7,9	-7,7	-0,5	-0,4	+12,8	7	
8		+2,1	+5,6	-4,0	-0,2	-9,2	-7,6	8	
9		-1,4	+2,9	-1,3	+2,3	+5,6	-0,5	9	
10	9	-0,4	+1,2	+0,1	-1,2	+0,3	+0,2	10	9
. . .	. . .	. . . .	. . . . .	. . . . .	. . . .	. . . . . .	. . . . .	. .	.

Stencil #19:

		+2,2
	+7,5	+1.0
-48,3	+1.0	
+1.0		

(s,-1,1) = - 38,6

Stencil #20:

(s,i,j) | +0.5 e
| -1.0
| +0.5 e

Stencil #21:

(c,i,j) | -0.5 e
| +1.0
| -0.5 e

i	j
0	0
0	
1	
1	
2	
2	
3	
4	0

$\overline{W}$		ndz		ν		$d\nu/dE$	
cos	sin	cos	sin	cos	sin	cos	sin
-529,504/2	0,0		0,0	+264,752/2	0,0	-5,471/2	0,0
+5,471 E/2	0,0	-532,007 E/2	0,0	-5,471 E/2	0,0	0,0	0,0
+26,63	-45,93	+104,0	+85,4	-11,41	+31,38	+2,	-5,
+58,214 E	+32,895 E	-32,895 E	+57,957 E	-29,107 E	-16,448 E	-16,448 E	+29,107 E
-20,8	+3,3	-3,4	-11,4	+8,1	-1,3	-3,	-16,
		+0,773 E	-1,368 E				
+0,7	-1,0	+0,4	+0,6	-0,3	+0,3	+1,	+1,
+0,1				-0,1			
+0,2				-0,1			
-1,4	+1,7	+1,	0,	+0,5	-0,8	+2,	+1,
-11,2	-38,6	-28,	+6,	+5,4	+17,5	-25,	+8,
-48,4	+26,4	+150,	+74,	+29,8	+1,4	-1,	+13,
-317,6	-808,5	+1450,	-566,	+174,4	+444,0	+247,	-97,
-2,1	-39,9	+1,	+8,	+1,9	+17,9	+28,	-3,
-6,9	-8,9	+3,	-3,	+2,4	+3,1	+8,	-6,
+0,7	+0,1			-0,2			+1,
-0,1	+0,2				-0,1		
-4,0	-0,8	-4,	-3,	+1,8	+0,3	-1,	+3,
-186,7	+152,6	+117,	+139,	+98,6	-79,8	+71,	+87,
-1351,7	+1034,1	-6648,	-9371,	+975,7	-774,0	-89,	-112,
-5728,6	+5643,8	-5020,	-5083,	+2762,3	-2721,5	-3033,	-3079,
+15,2	+2,2	+124,	+135,	-5,1	-0,8	-2,	+11,
-0,3	-5,6	+2,	0,	-0,5	+1,6	+5,	+2,
+0,1				-0,1			
				+0,1			
+0,9	-15,4	-8	-1,	-0,1	+7,1	-9,	0,
-7,0	-94,2	-154,	-29,	+4,1	+72,4	-24,	+1,
-353,0	-915,6	+1337,	-479,	+210,3	+513,7	+345,	-141,
-660,4	-280,8	+142,	-385,	+267,8	+113,6	+190,	-448,
-3,7	+6,2	-7,	+10,	+1,2	-1,8	-5,	-3,
+3,9	+1,0		+1,	-1,0	-0,2	-1,	+4,
-0,1	-0,1						
. . . .	. . . .	. . . .	. . . .	. . . . .	. . . .	. . . .	. . .
-6,0	-1,8	-1,	+4,	+3,0	+0,9	-1,	+3,
-41,1	-14,1	+1107,	-2282,	+29,2	+8,2	0,	0,
-109,9	+70,3	-70,	-106,	+54,8	-34,8	-35,	-56,
-1,8	-4,4	+4,	+2,	+0,5	+1,7	+3,	-1,
+6,1	+0,1	0,	+2,	-1,6	-0,2	0,	+5,
-3,4	+5,9	-2,	-1,	+0,8	-1,3	-5,	-3,
-1,5	-3,2	+1,		+0,3	+0,6	+3,	-2,
-0,6	-0,1			+0,1			-1,
. . . .	. . . .	. . . .	. . . . .	. . . . .	. . . . .	. . . .	. . . .
-263,2	0,0	+2,7	0,0	+131,6	0,0	-1,4	0,0
+89,6 T	0,0	-8713,6 T	0,0	-89,6 T	0,0	0,0	0,0
+26,6	-70,8	+104,1	+60,4	-11,4	+31,1	+2,	-5,
+1907,0 T	+1077,6 T	-1077,6 T	+1898,5 T	-953,5 T	-538,8 T	-538,8 T	+953,5 T
-22,3	+6,0	-6,2	-12,9	+8,9	-2,7	-4,	-17,
		+25,3 T	-44,8 T				
+0,7	-1,0	+0,3	+0,6	-0,3	+0,3	+1,	+1,
+0,1				-0,1			

Sheet 18 (cont.):

		R					
		cos			sin		
i	j	h = - 1	0	+ 1	- 1	0	+ 1
0	0						
0		-6,957 E/2	+1,308 E/2	-6,957 E/2	+52,453 E/2	0,0	-52,453 E/2
1		+24,762	-1,480	-9,013	-0,654	-0,313	+3,987
1							
2		-25,49	+2,43	-0,33	+3,63	-0,33	-0,07
2							
3		+1,4	-0,1	+0,1	-1,1	+0,1	-0,1
4	0	+0,3			+0,0	0,0	
				+0,1	+0,1		-0,1
-3	1			+0,2			+0,4
-2		-0,1	+0,1	-1,5	+0,2	-0,3	+2,9
-1		-0,7	+0,6	-5,8	-0,4	+2,6	-27,2
0		+6,3	-1,3	+7,2	-24,4	-2,0	+45,6
1		-16,1	-1,4	+30,9	-71,9	+1,5	+56,5
2		+2,5	0,0	-2,5	+12,4	-0,9	-3,0
3		-6,7	+0,6	-0,1	-11,8	+1,1	-0,2
4	1	+0,6	-0,1		+0,5		
		+0,1		+0,1	+0,2		
-2	2			0,0			+0,4
-1		-0,1	+0,6	-6,3	+0,1	-0,1	+0,5
0		-1,2	-4,4	+48,5	-1,0	+3,2	-33,4
1		-137,7	-0,5	+143,0	+32,8	+2,9	-63,3
2		+38,6	-2,4	-13,1	-26,8	+1,0	+15,7
3		-6,5	+0,5	+0,8	+2,9	-0,1	-1,4
4		+4,3	-0,4	+0,2	-5,5	+0,5	-0,1
5		-0,1			+0,3		
6	2	-0,1			+0,1		
							+0,1
-1	3	0,0	+0,1	-0,8	0,0	0,0	+0,4
0		-0,2	+0,2	-1,4	-0,3	+0,9	-9,5
1		+5,1	-8,8	+88,4	-6,8	-5,2	+62,6
2		-7,1	-0,5	+12,0	-21,4	+0,6	+14,6
3		+17,5	-0,9	-8,2	+12,0	-0,9	-2,6
4		-2,5	+0,2	+0,5	-3,1	+0,3	+0,1
5		+3,6	-0,4	+0,1	+1,0	-0,1	
6		0,0					
		-0,1					
. . . .		. . . .	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .
3	9			+0,1			
4		-2,0	+0,2	-0,5	-1,3	+2,7	-27,3
5		+0,1	-0,1	+1,1	-0,1	0,0	+0,4
6		+0,1	+0,1	-0,8	+0,2	-0,1	+0,8
7		-0,4	0,0	+0,1	-0,1	+0,1	-0,7
8		+0,4	0,0	-0,1	-0,4	0,0	+0,3
9		0,0		-0,1	+0,4		-0,1
10		+0,1			-0,1		
. . . .		. . . .	. . . . . .	. . . . .	. . . . .	. . . .	. . . .

Stencil #22:

$$\frac{(C,-1,i,j)}{-\ e/6} \quad \frac{(C,0,i,j)}{-\ 1/6} \quad \frac{(C,1,i,j)}{-\ e/6} \quad \frac{(c,i,j)}{-\ 1/2}$$

i	j	u cos	u sin	du/dE cos	du/dE sin	b	cos b	sin b
0	0	+49,525/2	0,0	+0,7	0,0	0.0	+1.0	0.0
0		+1,308 E/2	0,0					
1		-26,97	+3,32	-3,6	-25,5	0.46118	-0.9704	+0.2415
1		-6,957 E	-52,453 E	-52,453 E	+6,957 E			
2		-5,2	+2,6	+5,	+10,	0.9224	+0.8835	-0.4685
2								
3		-0,2			+1,	0:384	-0.75	+0.67
4	0	+0,1						
-3	1		+0,1					
-2		-0,4	-0,3	+1,	-1,	0.454	-0.96	+0.285
-1		+5,4	-18,9	+27,	+8,	0.9150	+0.861	-0.509
0		-23,2	-101,1	+45,	-10,	0.3762	-0.712	+0.702
1		+8,3	+59,4	+33,	-5,	0.83733	+0.5216	-0.8532
2		+24,2	+43,8	+68,	-38,	0.2985	-0.300	+0.954
3		-1,3	-1,4	-4,	+3,	0.760	+0.063	-0.998
4	1	-0,1	-0,1	0,	0,	0.221	+0.18	+0.98
-2	2							
-1		-0,6	-0,7	+1,	-1,	0.291	-0,265	+0.967
0		-148,4	+36,5	-32,	-131,	0.7523	+0.0145	-0.9999
1		+86,6	-57,3	-7,	-10,	0.213482	+0.22744	+0.97379
2		+134,1	-59,3	-66,	-150,	0.674664	-0.45587	-0.89004
3		-8,3	+10,1	+21,	+17,	0.1358	+0.6575	+0.7535
4		+0,3	-0,6	-2,	-1,	0.597	-0.82	-0.57
5		+0,1						
6	2							
-1	3	-0,1	-0,2					
0		+4,5	-5,5	+7,	+6,	0.128	+0.69	+0.72
1		-17,3	-36,1	+12,	-6,	0.5896	-0.846	-0.534
2		+105,4	+75,2	+51,	-71,	0.05081	+0.9495	+0.3139
3		+8,6	+10,6	+18,	-14,	0.51200	-0.9972	-0.0753
4		-4,4	-1,3	-3,	+12,	0.973	+0.986	-0.169
5		+0,1			-1,	0.434	-0.915	+0.403
6	3							
. . .		. . .	. . .	. . .	. . .	. . .	. . .	. . .
3	9	-2,0	-1,3	+1,	-2,	0.769	+0.12	-0.99
4		+0,4	+2,6	0,	0,	0.23008	+0.1248	+0.9922
5		-0,5	-27,1	-28,	+1,	0.6913	-0.3605	-0.9328
6		+0,8	+0,2	0,	-2,	0.152	+0.58	+0.82
7		-0,4	+0,5	+1,	+1,	0.614	-0.75	-0.66
8		+0,1	-0,3	-1,	0,	0.075	+0.89	+0.45
9			+0,1	+1,		0.536	-0.97	-0.22
10	9							
. . .		. . .	. . .	. . .	. . .	. . .	. . .	. . .
0	0	+22,3		+1,0				
0		+21,4 T						
1		-27,0	+3,4	-3,6	-25,5			
1		-227,9 T	-1718,2 T	-1718,2 T	+227,9 T			
2		-2,7	+2,2	+5,	+8,			
2								
3		-0,2			+1,			
4	0	+0,1						

Sheet 19:

		$(n\,dz)^{\circ}\,10^4$		$\nu\ 10^6$		$u\ 10^6$			
i	j	cos	sin	cos	sin	cos	sin	a	b
0	0	0	0	+782	0	+6	0	0.0	0.0
0		0	0	-90 T	0	+21 T	0		
1		+4246	-3044	+2687	+3685	+142	+138	+1.0	0.461182
1		-617 T	+1088 T	-954 T	-539 T	-228 T	-1718 T		
2		-102	+65	+9	-3	-3	+2	+2.0	0.9224
2	0	+14 T	-26 T						
-1	1	-16	+3	+5	+18	+5	-19	-1.442711	0.915
0		+86	+42	+30	+1	-23	-101	-0.442711	0.3762
1		+831	-324	+174	+444	+8	+59	+0.557289	0.83733
2		+1	+5	+2	+18	+24	+44	+1.557289	0.2985
3	1	+2	-2	+2	+3	-1	-1	+2.557289	0.760
-1	2	-2	-1	+2	0	-1	-1	-1.885422	0.291
0		+67	+80	+99	-80	-148	+36	-0.885422	0.7523
1		-3809	-5369	+976	-774	+87	-57	+0.114578	0.213482
2		-2876	-2913	+2762	-2722	+134	-59	+1.114578	0.674664
3	2	+71	+77	-5	-1	-8	+10	+2.114578	0.1358
0	3	-5	-1	0	+7	+4	-6	-1.328133	0.128
1		-88	-17	+4	+72	-17	-36	-0.328133	0.5896
2		+766	-274	+210	+514	+105	+75	+0.671867	0.05081
3		+82	-221	+268	+114	+9	+11	+1.671867	0.51200
4	3	-4	+6	+1	-2	-4	-1	+2.671867	0.973
1	4	+2	+10	+12	+2	-15	-5	-0.770844	0.966
2		+232	-390	+129	+32	+20	-3	+0.229156	0.42696
3		-247	-277	+280	-235	+23	-87	+1.229156	0.88815
4	4	+62	+8	-1	+75	-2	+4	+2.229156	0.3493
1	5	-2	-1	-1	+2	+2	0	-1.213555	0.342
2		-11	-49	-14	+13	-2	-8	-0.213555	0.8031
3		+180	+12	-3	+136	+33	+32	+0.786445	0.2643
4		+18	-43	+49	+28	+13	-1	+1.786445	0.7255
5	5	+6	+17	-23	+9	-2	0	+2.786445	0.1867
2	6	-1	+1	+1	+2	-2	-2	-0.65627	0.179
3		+49	-29	+15	+16	+5	+1	+0.34373	0.6404
4		-17	-39	+42	-14	+6	-15	+1.34373	0.1016
5		+14	+2	-3	+18	+2	+4	+2.34373	0.5628
6	6	-5	+4	-7	-7	0	-1	+3.34373	0.024
3	7	+28	-68	-11	+1	0	-3	-0.09898	0.0166
4		+56	+34	-27	+46	+9	+18	+0.90102	0.4778
5		+5	-7	+6	+8	+4	0	+1.90102	0.939
6		+1	+5	-7	+2	-1	+1	+2.90102	0.400
7	7	-2	-1	+1	-4	0	0	+3.90102	0.861
4	8	+9	0	+1	+4	+1	+1	+0.4583	0.854
5		0	-6	+6	+1	-2	-2	+1.4583	0.315
6		+3	+1	-2	+3	+1	+1	+2.4583	0.776
7	8	-2	+1	-2	-3	-1	0	+3.4583	0.237
3	9	-1	+2	+3	+1	-2	-1	-0.98440	0.769
4		+634	-1307	+29	+8	0	+3	+0.015601	0.23008
5		-40	-61	+55	-35	-1	-27	+1.015601	0.6913
6		+2	+1	0	+2	+1	0	+2.015601	0.152
7	9	0	+1	-2	0	0	0	+3.015601	0.614

Numerous terms of only one unit have been omitted. T = 0.0001 (JD - 2428000.5).

Stencil #1:

(c,j+h,j)		(s,j-h,j)	
+	+23,161	+	
	-	+53,123	+
	-	+53,961	-
+	-39,528	-	
	(c,j-h,j)		(s,j+h,j)

Stencil #2:

(C,-h-j,j)		(S,h-j,j) = 0	
+	+144,227	+	
	+	+40,839	-
	+	-40,839	+
+	+144,228	-	
	(C,h-j,j) = 0		(S,-h-j,j)

(c,2,1) = -16,4; (c,0,1) = +62,69; (s,0,1) = -107,08; (s,2,1) = -0,8; (C,-2,1) = +288,5; (S,-2,1) = -81,7.

Stencil #3:

-1.1225916	+0.1083386/2	-1.1225916	. . .	. . . .
-0,1079	-2,2367	-30,9387	-2,2367	-0,1079

(γ,0,1) = + 34,6103

Stencil #6:

Cos

+1,1	0,	0,	-1,2
+15,2	0,	+1,	+3,3
+144,2	-5,	-1,	-40,8
+583,5	+7,0	-22,6	-1403,1

+289,73 +1.0000039

+583,5	-20,31	-1,93	+1403,1
+144,2	+1,7	-3,6	+40,8
+15,2	+1,	0,	-3,3
+1,1			+1,2

(γ,1,1) = + 289,75

Stencil #7:

Sin

+1,2	0,	0,	+1,1
-3,3	0,	+1,	+15,2
+40,8	-5,	-1,	+144,2
+1403,1	+7,0	-22,6	+583,5

+1.0000039 +695,03

-1403,1	-20,31	-1,93	+583,5
-40,8	+1,7	-3,6	+144,2
+3,3	+1,0	0,	+15,2
-1,2			+1,1

(σ,1,1) = + 695,06

Stencil #8:

+127,50 -0.166667

		-19,4	-0.0010079		
		-116,90	+0.0428946		
+19,4	+81,383	+695,06	+0.8966335	+21,0	-6,0
+0.0007879	-0.0653095	-9,7	+0.0428946	-0.0653095	+0.0007879
		-16,	-0.0010079		

-885,562 +0.5

(σ,1,1) = + 147,11

Stencil #9:

(i, j)				
(i,0)				-0.02412691
(i,1)	-0.00000003	-0.00000468	-0.00029100	0.0
(i,2)	-0.00000023	-0.00000936	0.0	+0.02409883
(i,3)	-0.00000038	0.0	+0.00087164	+0.04812753
(i,4)	0.0	+0.00003736	+0.00232121	+0.07204418
(i,5)	+0.00000176	+0.00011661	+0.00434467	+0.09580712
(i,6)	+0.00000607	+0.00025148	+0.00693663	+0.11937496
(i,7)	+0.00001445	+0.00045553	+0.01009038	+0.14270672
(i,8)	+0.000029	+0.000742	+0.013798	+0.165762
. .	. . . . .	. . . . .	. . . . .	. . . . .

Stencil #10:

j = 1	j = 2	j = 3	j = 4	j = 5	j = 6
					+0.00001
	+0.000012	+0.000040	+0.000096	+0.00019	+0.00032
+0.000216	+0.000865	+0.001945	+0.003455	+0.00539	+0.00775
+0.020800	+0.041573	+0.062293	+0.082931	+0.10346	+0.12386
+0.999567	+0.998269	+0.996108	+0.993087	+0.98921	+0.98448
-0.020800	-0.041573	-0.062293	-0.082931	-0.10346	-0.12386
+0.000216	+0.000865	+0.001945	+0.003455	+0.00539	+0.00775
	-0.000012	-0.000040	-0.000096	-0.00019	-0.00032
					+0.00001

Stencil #11:

+577,772/2	0,0	+42,71	-116,90	+6,7	-22,6
-0.0367606	+0.1320350	+0.0035477	0.0	-0.0367606	-0.1320350

$(\gamma,0,1) = -18,35$

Stencil #13:

+0.015804	-2,
-0.671123	+3,6
+0.047413	-102,43
-0.001485	+673,2

-3,	+0.047413
+6,7	-1.004456
+75,4	-0.047413
-261,3	+0.004456

÷

+0.11457808

(C,-1,1,2) = - 173,8

Stencil #14:

+0.000743	-2,
-0.031608	+3,6
+1.004456	-102,43
-0.031608	+673,2
+0.000743	-6,

-3,	+0.002228
+6,7	-0.047413
+75,4	0.0
-261,3	+0.047413
+1,	-0.002228

÷

+0.11457808

(C,0,1,2) = - 1195,7

Stencil #15:

-0.001485	+3,6
+0.047413	-102,43
-0.671123	+673,2
+0.015804	-6,

+6,7	-0.004456
+75,4	+0.047413
-261,3	+1.004456
+1,	-0.047413

÷

+0.11457808

(C,1,1,2) = - 6246,6

j				
+1.0	-0.02412691			
+0.99941798	-0.04823978	+0.00087300	-0.00000936	+0.00000005
+0.99767292	-0.07229649	+0.00232662	-0.00004680	+0.00000069
+0.99476789	-0.09625507	+0.00435818	-0.00012624	+0.00000266
+0.99070795	-0.12007364	+0.00696363	-0.00026156	+0.00000722
+0.98550017	-0.14371068	+0.01013755	-0.00046645	+0.00001584
+0.97915366	-0.16712494	+0.01387325	-0.00075444	+0.00003037
+0.97167947	-0.19027562	+0.01816268	-0.00113881	+0.00005297
+0.963091	-0.213122	+0.022996	-0.001633	+0.000086
. . . . .	. . . . .	. . . . .	. . . . .	. . . . .

j = 7	j = 8	j = 9	j = 10	j = 11
+0.00002	+0.00003	+0.00005	+0.0001	+0.0001
+0.00051	+0.00076	+0.00108	+0.0015	+0.0020
+0.01053	+0.01372	+0.01733	+0.0213	+0.0257
+0.14409	+0.16414	+0.18398	+0.2036	+0.2229
+0.97890	+0.97249	+0.96525	+0.9572	+0.9483
-0.14409	-0.16414	-0.18398	-0.2036	-0.2229
+0.01053	+0.01372	+0.01733	+0.0213	+0.0257
-0.00051	-0.00076	-0.00108	-0.0015	-0.0020
+0.00002	+0.00003	+0.00005	+0.0001	+0.0001

Stencil #12:

+289,75	+695,06	+139,3	+21,0	+22,9	-6,0
-0.1320350	-0.0367606	0.0	+0.0035477	+0.1320350	-0.0367606

$(\sigma,1,2)$ = - 60,5

Stencil #16:

+1,	-0.015804
+2,6	+0.671123
-74,93	-0.047413
+664,8	+0.001485

+0.047413	-1,
-1.004456	-6,9
-0.047413	-42,2
+0.004456	+257,1

÷

+0.11457808

(S,-1,1,2) = +142,3

Stencil #17:

+1,	-0.000743
+2,6	+0.031608
-74,93	-1.004456
+664,8	+0.031608
+26,	-0.000743

+0.002228	-1,
-0.047413	-6,9
0.0	-42,2
+0.047413	+257,1
-0.002228	+9,

÷

+0.11457808

(S,0,1,2) = +949,9

Stencil #18:

+2,6	+0.001485
-74,93	-0.047413
+664,8	+0.671123
+26,	-0.015804

-0.004456	-6,9
+0.047413	-42,2
+1.004456	+257,1
-0.047413	+9,

÷

+0.11457808

(S,1,1,2) = +6154,4

The computed examples which illustrate the use of the stencils do not always agree exactly with the end figures of the same quantities as copied from the computing sheets. This is because the computing sheets contained "high" and "low" dots to indicate rounding off errors of nearly one half unit in the last place, and these were used as five units in the next place in the subsequent computations.

Solution for constants of integration:

$$\begin{array}{rlll}
(C - g_0) & + 0.251438\, k_1 & + 0.991159\, k_2 & = +4273, \\
k_0 & - 0.970403\, k_1 & + 0.241492\, k_2 & = +2624, \\
- k_0 & + 0.704305\, k_1 & - 0.181119\, k_2 & = -1629, \\
 & - 0.241492\, k_1\,(l_1) & + 0.970403\, k_2\,(l_2) & = +8394, \quad (-171,) \\
 & - 1.064391\, k_1\,(l_1) & + 0.241492\, k_2\,(l_2) & = +3980, \quad (-147,) \\
D = & + 1.091206 & &
\end{array}
\qquad
\begin{array}{rcr}
(C - g_0) & = & +12872{,}4 \\
k_0 & = & -848{,}7 \\
k_1 & = & -5397, \\
k_2 & = & -7307, \\
l_1 & = & +168{,}6 \\
l_2 & = & +134{,}3
\end{array}$$

$$M = +165\overset{\circ}{.}46244 + 0\overset{\circ}{.}18757559\,(\text{JD} - 2428000.5)$$

$$E = M + 5\overset{\circ}{.}38511 \sin E, \qquad i\,E - j\,\phi = \frac{E - 166.02537}{360}\,a + b$$

A computed position is determined by means of ((8,5)), using the vectorial constants shown on Sheet 1, page 130. A little consideration will make it evident that the mean motion is now only weakly determined; and it may be necessary to correct it by comparison with observations over as long an arc as they may be available. This will increase the accuracy of the predicted places in the future.

Table I gives A and ΔZ in units of the 7th decimal place, as a function of the astronomical latitude. This is followed by a collection of the values of the longitude, latitude, A, and ΔZ for some of the principal observatories of the world. Others may be added in the blank space provided at the bottom. See page 21.

Table II is a 4-decimal critical table of the Everett Second Difference Coefficients. When the interpolating fraction, n, is located in either of the two columns on the left, the respondent in the central column is E_0; when it is located in either column on the right, the respondent is E_1. This arrangement has been chosen so that the two places at which the table must be entered shall be as close together as possible. These coefficients will be needed to interpolate the condensed Tables III and IV, and the rectangular coordinates. See pages 12 and 24.

Table III gives $\eta\zeta$ and $\bar{y}$ for parabolic orbits, with the argument $\eta = \frac{2k(t_j - t_i)}{(r_i + r_j)^{3/2}}$. It also gives $\bar{y}$ and $\Delta\bar{y}_0$ with the argument $h = \frac{m^2}{1 + 5/6 + \xi}$. This is the solution of ⟨⟨5,19⟩⟩. In the latter case, $\bar{y} = \bar{y}_0 - \Delta\bar{y}_0$, where $\bar{y}_0 = [6 + 5\sqrt{1 + 44h/9}]/11$. See pages 56, 57, 67.

Table IV gives the ξ which appears in h above, and the Q's which are needed in Lambert's equation ⟨⟨5,45⟩⟩, both with the argument x, and separately for the ellipse and the hyperbola. In the case of Lambert's equation, $x = \frac{r_i + r_j + s}{4a}$ and $\frac{r_i + r_j - s}{4a}$. It also gives f(q), which is needed in Encke's method of special perturbations. This is equivalent to Table XI of Planetary Coordinates. See pages 66 and 98.

Table V gives the functions B, C, D, with the argument A, which are needed in nearly parabolic orbits. See page 37.

Table VI gives the coefficients which are needed in the interpolating formulas ⟨⟨1,20⟩⟩ and ⟨⟨1,21⟩⟩, with the interpolating fraction, n, as the argument. The adopted notation is as follows:

$$\begin{aligned}\iint^{t_0 + nh} f(t)\,dt^2 &= m^{ii}f_0 + {}''E_0 f_0 + {}''E_0^{ii}\Delta_0^{ii} + {}''E_0^{iv}\Delta_0^{iv} + \dots \\ &\quad + n^{ii}f_1 + {}''E_1 f_1 + {}''E_1^{ii}\Delta_1^{ii} + {}''E_1^{iv}\Delta_1^{iv} + \dots \\ \int^{t_0 + nh} f(t)\,dt &= {}^{i}f_{1/2} + {}'E_0 f_0 + {}'E_0^{ii}\Delta_0^{ii} + {}'E_0^{iv}\Delta_0^{iv} + \dots \\ &\quad + {}'E_1 f_1 + {}'E_1^{ii}\Delta_1^{ii} + {}'E_1^{iv}\Delta_1^{iv} + \dots\end{aligned}$$

These formulas correspond to Everett's formula for ordinary interpolation and possess all of its advantages. As explained on page 9, it is necessary for the tabulated functions to be multiplied by h in the case of a single integral and h^2 in the case of a double integration. If the single integral is taken from a table of double integration, it must be divided by h.

The signs are to be read on the left or right of the functions according as the argument is in the left or right hand column of the page. For convenience, the arguments are reproduced in the central columns also. The signs of the first differences must be determined by inspection.

Table VII is an "optimum interval" table which gives $1/r^3$ with the argument r^2. The interpolating formula is

$$F(r^2) = F_0 - N(D_1 - ND_2),$$

where N consists of all of the portion of r^2 which is not printed in full-size type in the r^2 column. The small-size figures in the r^2 column are to be used only in determining the appropriate interval within which the interpolation is to be made. The small-size figure at the end of F_0 is to be entered into the machine as the last figure of F_0 and then this place is to be completely dropped, and not rounded; this small-size end figure already contains an extra 5 units for the purpose of rounding.

This is the first time that such a table has ever been printed for use with a hand calculating machine. However, the computer will find that it is easier to use than an ordinary table which requires second difference interpolation, since no interpolating coefficients are needed; and it is not necessary to pay any attention to the irregular intervals. This table is equivalent to Table X of Planetary Coordinates, and it is designed to give at least seven significant figures over the whole range of the table, with a mean error of less than one unit in the last place.

The following examples illustrate its use:

$r^2 = 4.043172$, $F = (0.04686041 - 0.043172 \times 0.01420874) \times (-0.043172) + 0.124999851$
$= 0.12300327$

$r^2 = 14.143172$, $F = (0.002009212 - 0.043172 \times 0.000175938) \times (-0.043172) + 0.0188873632$
$= 0.018800949$

$r^2 = 39.982581$, $F = (0.000157648 - 0.982581 \times 0.000004714) \times (-0.982581) + 0.0041057817$
$= 0.003955431$

Table I

ø	A	ΔZ	ø	A	ΔZ	ø	A	ΔZ	ø	A	ΔZ
0	-427	0	15	-412	110	30	-370	212	45	-302	300
1	427	7	16	410	117	31	366	218	46	297	305
2	426	15	17	408	124	32	362	225	47	291	310
3	426	22	18	406	131	33	358	231	48	286	316
4	426	30	19	404	138	34	354	237	49	280	320
5	-425	37	20	-401	145	35	-350	243	50	-275	325
6	424	44	21	398	152	36	346	249	51	269	330
7	423	52	22	396	159	37	341	255	52	263	335
8	423	59	23	393	166	38	337	261	53	257	339
9	421	66	24	390	172	39	332	267	54	251	344
10	-420	74	25	-387	179	40	-327	273	55	-245	348
11	419	81	26	384	186	41	322	278	56	239	352
12	417	88	27	380	193	42	318	284	57	233	356
13	416	95	28	377	199	43	313	289	58	227	360
14	414	103	29	373	206	44	307	295	59	220	364
15	-412	110	30	-370	212	45	-302	300	60	-214	368

Observatory	L	ø	A	ΔZ
Algiers	0.00843	+36.8	-342	-254
Allegheny	0.77772	+40.5	-325	-276
Ann Arbor	0.76742	+42.3	-316	-286
Barcelona	0.00590	+41.4	-320	-281
Belgrade	0.05699	+44.8	-303	-299
Bergedorf	0.02845	+53.5	-254	-341
Berkeley	0.66039	+37.9	-337	-260
Bucharest	0.07253	+47.5	-305	-297
Cleveland	0.77342	+41.5	-320	-281
Copenhagen	0.03494	+55.7	-241	-351
Cordoba	0.82167	-31.4	-364	+221
Flagstaff	0.68976	+35.2	-349	-245
Good Hope	0.05132	-33.9	-354	+237
Greenwich	0.00000	+51.5	-266	-332
Harvard	0.80242	+42.4	-316	-286
Heidelberg	0.02422	+49.4	-278	-322
Johannesburg	0.07798	-26.2	-383	+187
Lick	0.66210	+37.3	-340	-257
Mc Donald	0.71104	+30.7	-367	-216
Mt. Wilson	0.67206	+34.2	-353	-239
New Haven	0.79744	+41.3	-321	-280
Nice	0.02028	+43.7	-309	-293
Oak Ridge	0.80122	+42.5	-315	-287
San Fernando	0.98276	+36.5	-344	-252
Santiago	0.80365	-33.6	-356	+235
Simeis	0.09444	+44.4	-305	-297
Uccle	0.01211	+50.8	-270	-329
Washington	0.78593	+38.9	-332	-267
Yerkes	0.75401	+42.6	-315	-287

Table II

n for E_0		E	n for E_1	
0.00000	1.00000	−0.0000	0.00000	1.00000
0.00015	0.99970	−0.0001	0.00030	0.99985
0.00045	0.99910	−0.0002	0.00090	0.99955
0.00075	0.99850	−0.0003	0.00150	0.99925
0.00105	0.99790	−0.0004	0.00210	0.99895
0.00135	0.99730	−0.0005	0.00270	0.99865
0.00165	0.99670	−0.0006	0.00330	0.99835
0.00196	0.99610	−0.0007	0.00390	0.99804
0.00226	0.99550	−0.0008	0.00450	0.99774
0.00256	0.99490	−0.0009	0.00510	0.99744
0.00286	0.99430	−0.0010	0.00570	0.99714
0.00317	0.99370	−0.0011	0.00630	0.99683
0.00347	0.99310	−0.0012	0.00690	0.99653
0.00377	0.99250	−0.0013	0.00750	0.99623
0.00407	0.99190	−0.0014	0.00810	0.99593
0.00438	0.99130	−0.0015	0.00870	0.99562
0.00468	0.99070	−0.0016	0.00930	0.99532
0.00499	0.99010	−0.0017	0.00990	0.99501
0.00529	0.98950	−0.0018	0.01050	0.99471
0.00560	0.98890	−0.0019	0.01110	0.99440
0.00590	0.98830	−0.0020	0.01170	0.99410
0.00621	0.98770	−0.0021	0.01230	0.99379
0.00651	0.98710	−0.0022	0.01290	0.99349
0.00682	0.98650	−0.0023	0.01350	0.99318
0.00713	0.98590	−0.0024	0.01410	0.99287
0.00743	0.98530	−0.0025	0.01470	0.99257
0.00774	0.98470	−0.0026	0.01530	0.99226
0.00805	0.98410	−0.0027	0.01590	0.99195
0.00835	0.98350	−0.0028	0.01650	0.99165
0.00866	0.98289	−0.0029	0.01711	0.99134
0.00897	0.98229	−0.0030	0.01771	0.99103
0.00928	0.98169	−0.0031	0.01831	0.99072
0.00959	0.98109	−0.0032	0.01891	0.99041
0.00990	0.98049	−0.0033	0.01951	0.99010
0.01021	0.97989	−0.0034	0.02011	0.98979
0.01052	0.97929	−0.0035	0.02071	0.98948
0.01083	0.97869	−0.0036	0.02131	0.98917
0.01114	0.97809	−0.0037	0.02191	0.98886
0.01145	0.97749	−0.0038	0.02251	0.98855
0.01176	0.97689	−0.0039	0.02311	0.98824
0.01207	0.97629	−0.0040	0.02371	0.98793
0.01238	0.97569	−0.0041	0.02431	0.98762
0.01269	0.97508	−0.0042	0.02492	0.98731
0.01300	0.97448	−0.0043	0.02552	0.98700
0.01331	0.97388	−0.0044	0.02612	0.98669
0.01363	0.97328	−0.0045	0.02672	0.98637
0.01394	0.97268	−0.0046	0.02732	0.98606
0.01425	0.97208	−0.0047	0.02792	0.98575
0.01457	0.97148	−0.0048	0.02852	0.98543
0.01488	0.97088	−0.0049	0.02912	0.98512
0.01520	0.97027	−0.0050	0.02973	0.98480
0.01551	0.96967	−0.0051	0.03033	0.98449
0.01582	0.96907	−0.0052	0.03093	0.98418
0.01614	0.96847	−0.0053	0.03153	0.98386
0.01645	0.96787	−0.0054	0.03213	0.98355
0.01677	0.96726	−0.0055	0.03274	0.98323
0.01709	0.96666	−0.0056	0.03334	0.98291
0.01740	0.96606	−0.0057	0.03394	0.98260
0.01772	0.96546	−0.0058	0.03454	0.98228
0.01804	0.96486	−0.0059	0.03514	0.98196
0.01835	0.96425	−0.0060	0.03575	0.98165
0.01867	0.96365	−0.0061	0.03635	0.98133
0.01899	0.96305	−0.0062	0.03695	0.98101
0.01931	0.96245	−0.0063	0.03755	0.98069
0.01962	0.96184	−0.0064	0.03816	0.98038
0.01994	0.96124	−0.0065	0.03876	0.98006
0.02026	0.96064	−0.0066	0.03936	0.97974
0.02058	0.96004	−0.0067	0.03996	0.97942
0.02090	0.95943	−0.0068	0.04057	0.97910
0.02122	0.95883	−0.0069	0.04117	0.97878
0.02154	0.95823	−0.0070	0.04177	0.97846
0.02186	0.95762	−0.0071	0.04238	0.97814
0.02218	0.95702		0.04298	0.97782

n for E_0		E	n for E_1	
0.02186	0.95762	−0.0071	0.04238	0.97814
0.02218	0.95702	−0.0072	0.04298	0.97782
0.02250	0.95642	−0.0073	0.04358	0.97750
0.02283	0.95581	−0.0074	0.04419	0.97717
0.02315	0.95521	−0.0075	0.04479	0.97685
0.02347	0.95461	−0.0076	0.04539	0.97653
0.02379	0.95400	−0.0077	0.04600	0.97621
0.02412	0.95340	−0.0078	0.04660	0.97588
0.02444	0.95279	−0.0079	0.04721	0.97556
0.02476	0.95219	−0.0080	0.04781	0.97524
0.02509	0.95159	−0.0081	0.04841	0.97491
0.02541	0.95098	−0.0082	0.04902	0.97459
0.02574	0.95038	−0.0083	0.04962	0.97426
0.02606	0.94977	−0.0084	0.05023	0.97394
0.02639	0.94917	−0.0085	0.05083	0.97361
0.02671	0.94856	−0.0086	0.05144	0.97329
0.02704	0.94796	−0.0087	0.05204	0.97296
0.02736	0.94735	−0.0088	0.05265	0.97264
0.02769	0.94675	−0.0089	0.05325	0.97231
0.02802	0.94614	−0.0090	0.05386	0.97198
0.02834	0.94554	−0.0091	0.05446	0.97166
0.02867	0.94493	−0.0092	0.05507	0.97133
0.02900	0.94433	−0.0093	0.05567	0.97100
0.02933	0.94372	−0.0094	0.05628	0.97067
0.02966	0.94312	−0.0095	0.05688	0.97034
0.02999	0.94251	−0.0096	0.05749	0.97001
0.03031	0.94190	−0.0097	0.05810	0.96969
0.03064	0.94130	−0.0098	0.05870	0.96936
0.03097	0.94069	−0.0099	0.05931	0.96903
0.03130	0.94008	−0.0100	0.05992	0.96870
0.03164	0.93948	−0.0101	0.06052	0.96836
0.03197	0.93887	−0.0102	0.06113	0.96803
0.03230	0.93826	−0.0103	0.06174	0.96770
0.03263	0.93766	−0.0104	0.06234	0.96737
0.03296	0.93705	−0.0105	0.06295	0.96704
0.03329	0.93644	−0.0106	0.06356	0.96671
0.03363	0.93584	−0.0107	0.06416	0.96637
0.03396	0.93523	−0.0108	0.06477	0.96604
0.03429	0.93462	−0.0109	0.06538	0.96571
0.03463	0.93401	−0.0110	0.06599	0.96537
0.03496	0.93340	−0.0111	0.06660	0.96504
0.03530	0.93280	−0.0112	0.06720	0.96470
0.03563	0.93219	−0.0113	0.06781	0.96437
0.03597	0.93158	−0.0114	0.06842	0.96403
0.03630	0.93097	−0.0115	0.06903	0.96370
0.03664	0.93036	−0.0116	0.06964	0.96336
0.03698	0.92975	−0.0117	0.07025	0.96302
0.03731	0.92914	−0.0118	0.07086	0.96269
0.03765	0.92853	−0.0119	0.07147	0.96235
0.03799	0.92793	−0.0120	0.07207	0.96201
0.03833	0.92732	−0.0121	0.07268	0.96167
0.03866	0.92671	−0.0122	0.07329	0.96134
0.03900	0.92610	−0.0123	0.07390	0.96100
0.03934	0.92549	−0.0124	0.07451	0.96066
0.03968	0.92488	−0.0125	0.07512	0.96032
0.04002	0.92427	−0.0126	0.07573	0.95998
0.04036	0.92366	−0.0127	0.07634	0.95964
0.04070	0.92304	−0.0128	0.07696	0.95930
0.04104	0.92243	−0.0129	0.07757	0.95896
0.04138	0.92182	−0.0130	0.07818	0.95862
0.04173	0.92121	−0.0131	0.07879	0.95827
0.04207	0.92060	−0.0132	0.07940	0.95793
0.04241	0.91999	−0.0133	0.08001	0.95759
0.04275	0.91938	−0.0134	0.08062	0.95725
0.04310	0.91876	−0.0135	0.08124	0.95690
0.04344	0.91815	−0.0136	0.08185	0.95656
0.04378	0.91754	−0.0137	0.08246	0.95622
0.04413	0.91693	−0.0138	0.08307	0.95587
0.04447	0.91631	−0.0139	0.08369	0.95553
0.04482	0.91570	−0.0140	0.08430	0.95518
0.04516	0.91509	−0.0141	0.08491	0.95484
0.04551	0.91447	−0.0142	0.08553	0.95449
0.04586	0.91386		0.08614	0.95414

n for E_0		E	n for E_1	
0.04551	0.91447	−0.0142	0.08553	0.95449
0.04586	0.91386	−0.0143	0.08614	0.95414
0.04620	0.91325	−0.0144	0.08675	0.95380
0.04655	0.91263	−0.0145	0.08737	0.95345
0.04690	0.91202	−0.0146	0.08798	0.95310
0.04725	0.91140	−0.0147	0.08860	0.95275
0.04759	0.91079	−0.0148	0.08921	0.95241
0.04794	0.91018	−0.0149	0.08982	0.95206
0.04829	0.90956	−0.0150	0.09044	0.95171
0.04864	0.90894	−0.0151	0.09106	0.95136
0.04899	0.90833	−0.0152	0.09167	0.95101
0.04934	0.90771	−0.0153	0.09229	0.95066
0.04969	0.90710	−0.0154	0.09290	0.95031
0.05004	0.90648	−0.0155	0.09352	0.94996
0.05040	0.90587	−0.0156	0.09413	0.94960
0.05075	0.90525	−0.0157	0.09475	0.94925
0.05110	0.90463	−0.0158	0.09537	0.94890
0.05145	0.90402	−0.0159	0.09598	0.94855
0.05181	0.90340	−0.0160	0.09660	0.94819
0.05216	0.90278	−0.0161	0.09722	0.94784
0.05251	0.90216	−0.0162	0.09784	0.94749
0.05287	0.90155	−0.0163	0.09845	0.94713
0.05322	0.90093	−0.0164	0.09907	0.94678
0.05358	0.90031	−0.0165	0.09969	0.94642
0.05393	0.89969	−0.0166	0.10031	0.94607
0.05429	0.89907	−0.0167	0.10093	0.94571
0.05465	0.89845	−0.0168	0.10155	0.94535
0.05500	0.89783	−0.0169	0.10217	0.94500
0.05536	0.89721	−0.0170	0.10279	0.94464
0.05572	0.89659	−0.0171	0.10341	0.94428
0.05608	0.89597	−0.0172	0.10403	0.94392
0.05644	0.89535	−0.0173	0.10465	0.94356
0.05680	0.89473	−0.0174	0.10527	0.94320
0.05716	0.89411	−0.0175	0.10589	0.94284
0.05752	0.89349	−0.0176	0.10651	0.94248
0.05788	0.89287	−0.0177	0.10713	0.94212
0.05824	0.89225	−0.0178	0.10775	0.94176
0.05860	0.89163	−0.0179	0.10837	0.94140
0.05896	0.89101	−0.0180	0.10899	0.94104
0.05932	0.89038	−0.0181	0.10962	0.94068
0.05969	0.88976	−0.0182	0.11024	0.94031
0.06005	0.88914	−0.0183	0.11086	0.93995
0.06041	0.88851	−0.0184	0.11149	0.93959
0.06078	0.88789	−0.0185	0.11211	0.93922
0.06114	0.88727	−0.0186	0.11273	0.93886
0.06151	0.88664	−0.0187	0.11336	0.93849
0.06187	0.88602	−0.0188	0.11398	0.93813
0.06224	0.88539	−0.0189	0.11461	0.93776
0.06261	0.88477	−0.0190	0.11523	0.93739
0.06297	0.88415	−0.0191	0.11585	0.93703
0.06334	0.88352	−0.0192	0.11648	0.93666
0.06371	0.88289	−0.0193	0.11711	0.93629
0.06408	0.88227	−0.0194	0.11773	0.93592
0.06445	0.88164	−0.0195	0.11836	0.93555
0.06481	0.88102	−0.0196	0.11898	0.93519
0.06518	0.88039	−0.0197	0.11961	0.93482
0.06556	0.87976	−0.0198	0.12024	0.93444
0.06593	0.87913	−0.0199	0.12087	0.93407
0.06630	0.87851	−0.0200	0.12149	0.93370
0.06667	0.87788	−0.0201	0.12212	0.93333
0.06704	0.87725	−0.0202	0.12275	0.93296
0.06741	0.87662	−0.0203	0.12338	0.93259
0.06779	0.87599	−0.0204	0.12401	0.93221
0.06816	0.87536	−0.0205	0.12464	0.93184
0.06853	0.87473	−0.0206	0.12527	0.93147
0.06891	0.87410	−0.0207	0.12590	0.93109
0.06928	0.87347	−0.0208	0.12653	0.93072
0.06966	0.87284	−0.0209	0.12716	0.93034
0.07004	0.87221	−0.0210	0.12779	0.92996
0.07041	0.87158	−0.0211	0.12842	0.92959
0.07079	0.87095	−0.0212	0.12905	0.92921
0.07117	0.87032	−0.0213	0.12968	0.92883
0.07154	0.86969		0.13031	0.92846

Table II (cont'd)

n for E_0		E	n for E_1	
0.07154	0.86969	−0.0214	0.13031	0.92846
0.07192	0.86905	−0.0215	0.13095	0.92808
0.07230	0.86842	−0.0216	0.13158	0.92770
0.07268	0.86779	−0.0217	0.13221	0.92732
0.07306	0.86716	−0.0218	0.13284	0.92694
0.07344	0.86652	−0.0219	0.13348	0.92656
0.07382	0.86589	−0.0220	0.13411	0.92618
0.07420	0.86525	−0.0221	0.13475	0.92580
0.07459	0.86462	−0.0222	0.13538	0.92541
0.07497	0.86398	−0.0223	0.13602	0.92503
0.07535	0.86335	−0.0224	0.13665	0.92465
0.07574	0.86271	−0.0225	0.13729	0.92426
0.07612	0.86208	−0.0226	0.13792	0.92388
0.07651	0.86144	−0.0227	0.13856	0.92349
0.07689	0.86080	−0.0228	0.13920	0.92311
0.07728	0.86017	−0.0229	0.13983	0.92272
0.07766	0.85953	−0.0230	0.14047	0.92234
0.07805	0.85889	−0.0231	0.14111	0.92195
0.07844	0.85825	−0.0232	0.14175	0.92156
0.07883	0.85761	−0.0233	0.14239	0.92117
0.07921	0.85697	−0.0234	0.14303	0.92079
0.07960	0.85633	−0.0235	0.14367	0.92040
0.07999	0.85570	−0.0236	0.14430	0.92001
0.08038	0.85505	−0.0237	0.14495	0.91962
0.08077	0.85441	−0.0238	0.14559	0.91923
0.08116	0.85377	−0.0239	0.14623	0.91884
0.08156	0.85313	−0.0240	0.14687	0.91844
0.08195	0.85249	−0.0241	0.14751	0.91805
0.08234	0.85185	−0.0242	0.14815	0.91766
0.08273	0.85121	−0.0243	0.14879	0.91727
0.08313	0.85056	−0.0244	0.14944	0.91687
0.08352	0.84992	−0.0245	0.15008	0.91648
0.08392	0.84928	−0.0246	0.15072	0.91608
0.08431	0.84863	−0.0247	0.15137	0.91569
0.08471	0.84799	−0.0248	0.15201	0.91529
0.08511	0.84734	−0.0249	0.15266	0.91489
0.08550	0.84670	−0.0250	0.15330	0.91450
0.08590	0.84605	−0.0251	0.15395	0.91410
0.08630	0.84541	−0.0252	0.15459	0.91370
0.08670	0.84476	−0.0253	0.15524	0.91330
0.08710	0.84411	−0.0254	0.15589	0.91290
0.08750	0.84346	−0.0255	0.15654	0.91250
0.08790	0.84282	−0.0256	0.15718	0.91210
0.08830	0.84217	−0.0257	0.15783	0.91170
0.08870	0.84152	−0.0258	0.15848	0.91130
0.08911	0.84087	−0.0259	0.15913	0.91089
0.08951	0.84022	−0.0260	0.15978	0.91049
0.08991	0.83957	−0.0261	0.16043	0.91009
0.09032	0.83892	−0.0262	0.16108	0.90968
0.09072	0.83827	−0.0263	0.16173	0.90928
0.09113	0.83762	−0.0264	0.16238	0.90887
0.09153	0.83697	−0.0265	0.16303	0.90847
0.09194	0.83631	−0.0266	0.16369	0.90806
0.09235	0.83566	−0.0267	0.16434	0.90765
0.09276	0.83501	−0.0268	0.16499	0.90724
0.09316	0.83436	−0.0269	0.16564	0.90684
0.09357	0.83370	−0.0270	0.16630	0.90643
0.09398	0.83305	−0.0271	0.16695	0.90602
0.09439	0.83239	−0.0272	0.16761	0.90561
0.09481	0.83174	−0.0273	0.16826	0.90519
0.09522	0.83108	−0.0274	0.16892	0.90478
0.09563	0.83042	−0.0275	0.16958	0.90437
0.09604	0.82977	−0.0276	0.17023	0.90396
0.09646	0.82911	−0.0277	0.17089	0.90354
0.09687	0.82845	−0.0278	0.17155	0.90313
0.09729	0.82779	−0.0279	0.17221	0.90271
0.09770	0.82713	−0.0280	0.17287	0.90230
0.09812	0.82647	−0.0281	0.17353	0.90188
0.09854	0.82581	−0.0282	0.17419	0.90146
0.09895	0.82515	−0.0283	0.17485	0.90105
0.09937	0.82449	−0.0284	0.17551	0.90063
0.09979	0.82383	−0.0285	0.17617	0.90021
0.10020	0.82317		0.17683	0.89980

n for E_0		E	n for E_1	
0.09979	0.82383	−0.0285	0.17617	0.90021
0.10020	0.82317	−0.0286	0.17683	0.89980
0.10063	0.82251	−0.0287	0.17749	0.89937
0.10105	0.82185	−0.0288	0.17815	0.89895
0.10147	0.82118	−0.0289	0.17882	0.89853
0.10189	0.82052	−0.0290	0.17948	0.89811
0.10232	0.81985	−0.0291	0.18015	0.89768
0.10274	0.81919	−0.0292	0.18081	0.89726
0.10317	0.81852	−0.0293	0.18148	0.89683
0.10359	0.81786	−0.0294	0.18214	0.89641
0.10402	0.81719	−0.0295	0.18281	0.89598
0.10444	0.81652	−0.0296	0.18348	0.89556
0.10487	0.81586	−0.0297	0.18414	0.89513
0.10530	0.81519	−0.0298	0.18481	0.89470
0.10573	0.81452	−0.0299	0.18548	0.89427
0.10616	0.81385	−0.0300	0.18615	0.89384
0.10659	0.81318	−0.0301	0.18682	0.89341
0.10702	0.81251	−0.0302	0.18749	0.89298
0.10745	0.81184	−0.0303	0.18816	0.89255
0.10788	0.81117	−0.0304	0.18883	0.89212
0.10831	0.81049	−0.0305	0.18951	0.89169
0.10875	0.80982	−0.0306	0.19018	0.89125
0.10918	0.80915	−0.0307	0.19085	0.89082
0.10961	0.80847	−0.0308	0.19153	0.89039
0.11005	0.80780	−0.0309	0.19220	0.88995
0.11049	0.80712	−0.0310	0.19288	0.88951
0.11092	0.80645	−0.0311	0.19355	0.88908
0.11136	0.80577	−0.0312	0.19423	0.88864
0.11180	0.80510	−0.0313	0.19490	0.88820
0.11224	0.80442	−0.0314	0.19558	0.88776
0.11268	0.80374	−0.0315	0.19626	0.88732
0.11312	0.80306	−0.0316	0.19694	0.88688
0.11356	0.80238	−0.0317	0.19762	0.88644
0.11400	0.80170	−0.0318	0.19830	0.88600
0.11445	0.80102	−0.0319	0.19898	0.88555
0.11489	0.80034	−0.0320	0.19966	0.88511
0.11534	0.79966	−0.0321	0.20034	0.88466
0.11578	0.79898	−0.0322	0.20102	0.88422
0.11623	0.79829	−0.0323	0.20171	0.88377
0.11668	0.79761	−0.0324	0.20239	0.88332
0.11712	0.79693	−0.0325	0.20307	0.88288
0.11757	0.79624	−0.0326	0.20376	0.88243
0.11802	0.79555	−0.0327	0.20445	0.88198
0.11847	0.79487	−0.0328	0.20513	0.88153
0.11892	0.79418	−0.0329	0.20582	0.88108
0.11938	0.79349	−0.0330	0.20651	0.88062
0.11983	0.79281	−0.0331	0.20719	0.88017
0.12028	0.79212	−0.0332	0.20788	0.87972
0.12074	0.79143	−0.0333	0.20857	0.87926
0.12119	0.79074	−0.0334	0.20926	0.87881
0.12165	0.79004	−0.0335	0.20996	0.87835
0.12210	0.78935	−0.0336	0.21065	0.87790
0.12256	0.78866	−0.0337	0.21134	0.87744
0.12302	0.78797	−0.0338	0.21203	0.87698
0.12348	0.78727	−0.0339	0.21273	0.87652
0.12394	0.78658	−0.0340	0.21342	0.87606
0.12440	0.78588	−0.0341	0.21412	0.87560
0.12486	0.78519	−0.0342	0.21481	0.87514
0.12532	0.78449	−0.0343	0.21551	0.87468
0.12579	0.78379	−0.0344	0.21621	0.87421
0.12625	0.78310	−0.0345	0.21690	0.87375
0.12672	0.78240	−0.0346	0.21760	0.87328
0.12719	0.78170	−0.0347	0.21830	0.87281
0.12765	0.78100	−0.0348	0.21900	0.87235
0.12812	0.78029	−0.0349	0.21971	0.87188
0.12859	0.77959	−0.0350	0.22041	0.87141
0.12906	0.77889	−0.0351	0.22111	0.87094
0.12953	0.77819	−0.0352	0.22181	0.87047
0.13000	0.77748	−0.0353	0.22252	0.87000
0.13047	0.77678	−0.0354	0.22322	0.86953
0.13095	0.77607	−0.0355	0.22393	0.86905
0.13142	0.77537	−0.0356	0.22463	0.86858
0.13190	0.77466		0.22534	0.86810

n for E_0		E	n for E_1	
0.13142	0.77537	−0.0356	0.22463	0.86858
0.13190	0.77466	−0.0357	0.22534	0.86810
0.13237	0.77395	−0.0358	0.22605	0.86763
0.13285	0.77324	−0.0359	0.22676	0.86715
0.13333	0.77253	−0.0360	0.22747	0.86667
0.13381	0.77182	−0.0361	0.22818	0.86619
0.13429	0.77111	−0.0362	0.22889	0.86571
0.13477	0.77040	−0.0363	0.22960	0.86523
0.13525	0.76968	−0.0364	0.23032	0.86475
0.13574	0.76897	−0.0365	0.23103	0.86426
0.13622	0.76825	−0.0366	0.23175	0.86378
0.13671	0.76754	−0.0367	0.23246	0.86329
0.13719	0.76682	−0.0368	0.23318	0.86281
0.13768	0.76610	−0.0369	0.23390	0.86232
0.13817	0.76539	−0.0370	0.23461	0.86183
0.13866	0.76467	−0.0371	0.23533	0.86134
0.13914	0.76395	−0.0372	0.23605	0.86086
0.13964	0.76323	−0.0373	0.23677	0.86036
0.14013	0.76250	−0.0374	0.23750	0.85987
0.14062	0.76178	−0.0375	0.23822	0.85938
0.14111	0.76106	−0.0376	0.23894	0.85889
0.14161	0.76033	−0.0377	0.23967	0.85839
0.14211	0.75961	−0.0378	0.24039	0.85789
0.14260	0.75888	−0.0379	0.24112	0.85740
0.14310	0.75815	−0.0380	0.24185	0.85690
0.14360	0.75743	−0.0381	0.24257	0.85640
0.14410	0.75670	−0.0382	0.24330	0.85590
0.14460	0.75597	−0.0383	0.24403	0.85540
0.14511	0.75524	−0.0384	0.24476	0.85489
0.14561	0.75450	−0.0385	0.24550	0.85439
0.14612	0.75377	−0.0386	0.24623	0.85388
0.14662	0.75304	−0.0387	0.24696	0.85338
0.14713	0.75230	−0.0388	0.24770	0.85287
0.14764	0.75157	−0.0389	0.24843	0.85236
0.14814	0.75083	−0.0390	0.24917	0.85186
0.14866	0.75009	−0.0391	0.24991	0.85134
0.14917	0.74935	−0.0392	0.25065	0.85083
0.14968	0.74861	−0.0393	0.25139	0.85032
0.15019	0.74787	−0.0394	0.25213	0.84981
0.15071	0.74713	−0.0395	0.25287	0.84929
0.15122	0.74639	−0.0396	0.25361	0.84878
0.15174	0.74564	−0.0397	0.25436	0.84826
0.15226	0.74490	−0.0398	0.25510	0.84774
0.15278	0.74415	−0.0399	0.25585	0.84722
0.15330	0.74341	−0.0400	0.25659	0.84670
0.15382	0.74266	−0.0401	0.25734	0.84618
0.15434	0.74191	−0.0402	0.25809	0.84566
0.15487	0.74116	−0.0403	0.25884	0.84513
0.15540	0.74041	−0.0404	0.25959	0.84460
0.15592	0.73965	−0.0405	0.26035	0.84408
0.15645	0.73890	−0.0406	0.26110	0.84355
0.15698	0.73814	−0.0407	0.26186	0.84302
0.15751	0.73739	−0.0408	0.26261	0.84249
0.15804	0.73663	−0.0409	0.26337	0.84196
0.15858	0.73587	−0.0410	0.26413	0.84142
0.15911	0.73511	−0.0411	0.26489	0.84089
0.15965	0.73435	−0.0412	0.26565	0.84035
0.16018	0.73359	−0.0413	0.26641	0.83982
0.16072	0.73283	−0.0414	0.26717	0.83928
0.16126	0.73207	−0.0415	0.26793	0.83874
0.16180	0.73130	−0.0416	0.26870	0.83820
0.16234	0.73053	−0.0417	0.26947	0.83766
0.16289	0.72977	−0.0418	0.27023	0.83711
0.16343	0.72900	−0.0419	0.27100	0.83657
0.16398	0.72823	−0.0420	0.27177	0.83602
0.16453	0.72745	−0.0421	0.27255	0.83547
0.16508	0.72668	−0.0422	0.27332	0.83492
0.16563	0.72591	−0.0423	0.27409	0.83437
0.16618	0.72513	−0.0424	0.27487	0.83382
0.16673	0.72436	−0.0425	0.27564	0.83327
0.16729	0.72358	−0.0426	0.27642	0.83271
0.16784	0.72280	−0.0427	0.27720	0.83216
0.16840	0.72202		0.27798	0.83160

n for E_0		E	n for E_1	
0.16840	0.72202	−0.0428	0.27798	0.83160
0.16896	0.72124	−0.0429	0.27876	0.83104
0.16952	0.72045	−0.0430	0.27955	0.83048
0.17008	0.71967	−0.0431	0.28033	0.82992
0.17064	0.71888	−0.0432	0.28112	0.82936
0.17121	0.71810	−0.0433	0.28190	0.82879
0.17178	0.71731	−0.0434	0.28269	0.82822
0.17234	0.71652	−0.0435	0.28348	0.82766
0.17291	0.71573	−0.0436	0.28427	0.82709
0.17348	0.71493	−0.0437	0.28507	0.82652
0.17406	0.71414	−0.0438	0.28586	0.82594
0.17463	0.71335	−0.0439	0.28665	0.82537
0.17521	0.71255	−0.0440	0.28745	0.82479
0.17579	0.71175	−0.0441	0.28825	0.82421
0.17636	0.71095	−0.0442	0.28905	0.82364
0.17694	0.71015	−0.0443	0.28985	0.82306
0.17753	0.70935	−0.0444	0.29065	0.82247
0.17811	0.70854	−0.0445	0.29146	0.82189
0.17869	0.70773	−0.0446	0.29227	0.82131
0.17928	0.70693	−0.0447	0.29307	0.82072
0.17987	0.70612	−0.0448	0.29388	0.82013
0.18046	0.70531	−0.0449	0.29469	0.81954
0.18105	0.70450	−0.0450	0.29550	0.81895
0.18165	0.70368	−0.0451	0.29632	0.81835
0.18224	0.70287	−0.0452	0.29713	0.81776
0.18284	0.70205	−0.0453	0.29795	0.81716
0.18344	0.70123	−0.0454	0.29877	0.81656
0.18404	0.70041	−0.0455	0.29959	0.81596
0.18464	0.69959	−0.0456	0.30041	0.81536
0.18525	0.69877	−0.0457	0.30123	0.81475
0.18585	0.69794	−0.0458	0.30206	0.81415
0.18646	0.69711	−0.0459	0.30289	0.81354
0.18707	0.69628	−0.0460	0.30372	0.81293
0.18768	0.69545	−0.0461	0.30455	0.81232
0.18829	0.69462	−0.0462	0.30538	0.81171
0.18891	0.69379	−0.0463	0.30621	0.81109
0.18953	0.69295	−0.0464	0.30705	0.81047
0.19015	0.69211	−0.0465	0.30789	0.80985
0.19077	0.69128	−0.0466	0.30872	0.80923
0.19139	0.69043	−0.0467	0.30957	0.80861
0.19201	0.68959	−0.0468	0.31041	0.80799
0.19264	0.68875	−0.0469	0.31125	0.80736
0.19327	0.68790	−0.0470	0.31210	0.80673
0.19390	0.68705	−0.0471	0.31295	0.80610
0.19453	0.68620	−0.0472	0.31380	0.80547
0.19517	0.68535	−0.0473	0.31465	0.80483
0.19581	0.68449	−0.0474	0.31551	0.80419
0.19645	0.68364	−0.0475	0.31636	0.80355
0.19709	0.68278	−0.0476	0.31722	0.80291
0.19773	0.68192	−0.0477	0.31808	0.80227
0.19838	0.68106	−0.0478	0.31894	0.80162
0.19902	0.68019	−0.0479	0.31981	0.80098
0.19967	0.67932	−0.0480	0.32068	0.80033
0.20033	0.67846	−0.0481	0.32154	0.79967
0.20098	0.67758	−0.0482	0.32242	0.79902
0.20164	0.67671	−0.0483	0.32329	0.79836
0.20230	0.67584	−0.0484	0.32416	0.79770
0.20296	0.67496	−0.0485	0.32504	0.79704
0.20362	0.67408	−0.0486	0.32592	0.79638
0.20428	0.67320	−0.0487	0.32680	0.79572
0.20495	0.67231	−0.0488	0.32769	0.79505
0.20563	0.67143	−0.0489	0.32857	0.79437
0.20630	0.67054	−0.0490	0.32946	0.79370
0.20697	0.66965	−0.0491	0.33035	0.79303
0.20765	0.66875	−0.0492	0.33125	0.79235
0.20833	0.66786	−0.0493	0.33214	0.79167
0.20902	0.66696	−0.0494	0.33304	0.79098
0.20970	0.66606	−0.0495	0.33394	0.79030
0.21039	0.66516	−0.0496	0.33484	0.78961
0.21108	0.66425	−0.0497	0.33575	0.78892
0.21177	0.66334	−0.0498	0.33666	0.78823
0.21247	0.66243	−0.0499	0.33757	0.78753
0.21317	0.66152		0.33848	0.78683

n for E_0		E	n for E_1	
0.21247	0.66243	−0.0499	0.33757	0.78753
0.21317	0.66152	−0.0500	0.33848	0.78683
0.21387	0.66061	−0.0501	0.33939	0.78613
0.21457	0.65969	−0.0502	0.34031	0.78543
0.21528	0.65877	−0.0503	0.34123	0.78472
0.21599	0.65784	−0.0504	0.34216	0.78401
0.21670	0.65692	−0.0505	0.34308	0.78330
0.21742	0.65599	−0.0506	0.34401	0.78258
0.21813	0.65506	−0.0507	0.34494	0.78187
0.21885	0.65412	−0.0508	0.34588	0.78115
0.21958	0.65318	−0.0509	0.34682	0.78042
0.22031	0.65224	−0.0510	0.34776	0.77969
0.22104	0.65130	−0.0511	0.34870	0.77896
0.22177	0.65036	−0.0512	0.34964	0.77823
0.22250	0.64941	−0.0513	0.35059	0.77750
0.22324	0.64846	−0.0514	0.35154	0.77676
0.22399	0.64750	−0.0515	0.35250	0.77601
0.22473	0.64654	−0.0516	0.35346	0.77527
0.22548	0.64558	−0.0517	0.35442	0.77452
0.22623	0.64462	−0.0518	0.35538	0.77377
0.22699	0.64365	−0.0519	0.35635	0.77301
0.22775	0.64268	−0.0520	0.35732	0.77225
0.22851	0.64170	−0.0521	0.35830	0.77149
0.22927	0.64072	−0.0522	0.35928	0.77073
0.23004	0.63974	−0.0523	0.36026	0.76996
0.23081	0.63876	−0.0524	0.36124	0.76919
0.23159	0.63777	−0.0525	0.36223	0.76841
0.23237	0.63678	−0.0526	0.36322	0.76763
0.23315	0.63579	−0.0527	0.36421	0.76685
0.23394	0.63479	−0.0528	0.36521	0.76606
0.23473	0.63379	−0.0529	0.36621	0.76527
0.23553	0.63278	−0.0530	0.36722	0.76447
0.23633	0.63177	−0.0531	0.36823	0.76367
0.23713	0.63076	−0.0532	0.36924	0.76287
0.23793	0.62974	−0.0533	0.37026	0.76207
0.23874	0.62872	−0.0534	0.37128	0.76126
0.23956	0.62769	−0.0535	0.37231	0.76044
0.24038	0.62666	−0.0536	0.37334	0.75962
0.24120	0.62563	−0.0537	0.37437	0.75880
0.24203	0.62459	−0.0538	0.37541	0.75797
0.24286	0.62355	−0.0539	0.37645	0.75714
0.24369	0.62251	−0.0540	0.37749	0.75631
0.24453	0.62146	−0.0541	0.37854	0.75547
0.24538	0.62040	−0.0542	0.37960	0.75462
0.24623	0.61934	−0.0543	0.38066	0.75377
0.24708	0.61828	−0.0544	0.38172	0.75292
0.24794	0.61721	−0.0545	0.38279	0.75206
0.24881	0.61614	−0.0546	0.38386	0.75119
0.24967	0.61506	−0.0547	0.38494	0.75033
0.25055	0.61398	−0.0548	0.38602	0.74945
0.25142	0.61289	−0.0549	0.38711	0.74858
0.25231	0.61180	−0.0550	0.38820	0.74769
0.25320	0.61070	−0.0551	0.38930	0.74680
0.25409	0.60960	−0.0552	0.39040	0.74591
0.25499	0.60849	−0.0553	0.39151	0.74501
0.25589	0.60738	−0.0554	0.39262	0.74411
0.25681	0.60626	−0.0555	0.39374	0.74319
0.25772	0.60513	−0.0556	0.39487	0.74228
0.25864	0.60400	−0.0557	0.39600	0.74136
0.25957	0.60287	−0.0558	0.39713	0.74043
0.26050	0.60173	−0.0559	0.39827	0.73950
0.26144	0.60058	−0.0560	0.39942	0.73856
0.26239	0.59942	−0.0561	0.40058	0.73761
0.26334	0.59826	−0.0562	0.40174	0.73666
0.26430	0.59710	−0.0563	0.40290	0.73570
0.26527	0.59592	−0.0564	0.40408	0.73473
0.26624	0.59474	−0.0565	0.40526	0.73376
0.26722	0.59356	−0.0566	0.40644	0.73278
0.26820	0.59236	−0.0567	0.40764	0.73180
0.26920	0.59116	−0.0568	0.40884	0.73080
0.27020	0.58996	−0.0569	0.41004	0.72980
0.27120	0.58874	−0.0570	0.41126	0.72880
0.27222	0.58752		0.41248	0.72778

n for E_0		E	n for E_1	
0.27120	0.58874	−0.0570	0.41126	0.72880
0.27222	0.58752	−0.0571	0.41248	0.72778
0.27324	0.58629	−0.0572	0.41371	0.72676
0.27427	0.58506	−0.0573	0.41494	0.72573
0.27531	0.58381	−0.0574	0.41619	0.72469
0.27636	0.58256	−0.0575	0.41744	0.72364
0.27741	0.58130	−0.0576	0.41870	0.72259
0.27847	0.58002	−0.0577	0.41998	0.72153
0.27955	0.57874	−0.0578	0.42126	0.72045
0.28063	0.57746	−0.0579	0.42254	0.71937
0.28172	0.57616	−0.0580	0.42384	0.71828
0.28282	0.57486	−0.0581	0.42514	0.71718
0.28393	0.57354	−0.0582	0.42646	0.71607
0.28505	0.57222	−0.0583	0.42778	0.71495
0.28618	0.57088	−0.0584	0.42912	0.71382
0.28732	0.56953	−0.0585	0.43047	0.71268
0.28847	0.56818	−0.0586	0.43182	0.71153
0.28963	0.56681	−0.0587	0.43319	0.71037
0.29081	0.56543	−0.0588	0.43457	0.70919
0.29199	0.56404	−0.0589	0.43596	0.70801
0.29319	0.56264	−0.0590	0.43736	0.70681
0.29440	0.56123	−0.0591	0.43877	0.70560
0.29562	0.55980	−0.0592	0.44020	0.70438
0.29685	0.55836	−0.0593	0.44164	0.70315
0.29810	0.55691	−0.0594	0.44309	0.70190
0.29936	0.55544	−0.0595	0.44456	0.70064
0.30064	0.55396	−0.0596	0.44604	0.69936
0.30193	0.55246	−0.0597	0.44754	0.69807
0.30324	0.55095	−0.0598	0.44905	0.69676
0.30456	0.54943	−0.0599	0.45057	0.69544
0.30590	0.54788	−0.0600	0.45212	0.69410
0.30726	0.54632	−0.0601	0.45368	0.69274
0.30863	0.54475	−0.0602	0.45525	0.69137
0.31002	0.54315	−0.0603	0.45685	0.68998
0.31143	0.54154	−0.0604	0.45846	0.68857
0.31286	0.53990	−0.0605	0.46010	0.68714
0.31431	0.53825	−0.0606	0.46175	0.68569
0.31579	0.53657	−0.0607	0.46343	0.68421
0.31728	0.53487	−0.0608	0.46513	0.68272
0.31880	0.53315	−0.0609	0.46685	0.68120
0.32034	0.53141	−0.0610	0.46859	0.67966
0.32191	0.52963	−0.0611	0.47037	0.67809
0.32350	0.52784	−0.0612	0.47216	0.67650
0.32513	0.52601	−0.0613	0.47399	0.67487
0.32678	0.52416	−0.0614	0.47584	0.67322
0.32846	0.52227	−0.0615	0.47773	0.67154
0.33018	0.52035	−0.0616	0.47965	0.66982
0.33193	0.51839	−0.0617	0.48161	0.66807
0.33372	0.51640	−0.0618	0.48360	0.66628
0.33555	0.51437	−0.0619	0.48563	0.66445
0.33742	0.51230	−0.0620	0.48770	0.66258
0.33934	0.51018	−0.0621	0.48982	0.66066
0.34130	0.50802	−0.0622	0.49198	0.65870
0.34332	0.50580	−0.0623	0.49420	0.65668
0.34539	0.50353	−0.0624	0.49647	0.65461
0.34752	0.50119	−0.0625	0.49881	0.65248
0.34972	0.49879	−0.0626	0.50121	0.65028
0.35199	0.49632	−0.0627	0.50368	0.64801
0.35434	0.49377	−0.0628	0.50623	0.64566
0.35678	0.49112	−0.0629	0.50888	0.64322
0.35932	0.48838	−0.0630	0.51162	0.64068
0.36197	0.48553	−0.0631	0.51447	0.63803
0.36475	0.48255	−0.0632	0.51745	0.63525
0.36768	0.47943	−0.0633	0.52057	0.63232
0.37078	0.47613	−0.0634	0.52387	0.62922
0.37408	0.47262	−0.0635	0.52738	0.62592
0.37764	0.46886	−0.0636	0.53114	0.62236
0.38152	0.46479	−0.0637	0.53521	0.61848
0.38581	0.46029	−0.0638	0.53971	0.61419
0.39070	0.45519	−0.0639	0.54481	0.60930
0.39652	0.44918	−0.0640	0.55082	0.60348
0.40413	0.44137	−0.0641	0.55863	0.59587
0.44137	0.40413		0.59587	0.55863

Table III

η	$\eta\zeta$			$\bar{y}$			h	$\bar{y}$			$\Delta\bar{y}_0\ 10^8$		
0.00	0.000 0000	10 0000	0	1.000 0000	333	666	0.00	1.000 0000	10 9785	−2710	0	1	0
0.01	0.010 0000	10 0003	3	1.000 0333	1001	668	0.01	1.010 9785	10 7255	−2530	1	9	8
0.02	0.020 0003	10 0008	5	1.000 1334	1668	667	0.02	1.021 7040	10 4890	−2365	10	20	11
0.03	0.030 0011	10 0016	8	1.000 3002	2338	670	0.03	1.032 1930	10 2675	−2215	30	37	17
0.04	0.040 0027	10 0025	9	1.000 5340	3011	673	0.04	1.042 4605	10 0595	−2080	67	57	20
0.05	0.050 0052	10 0038	13	1.000 8351	3685	674	0.05	1.052 5200	9 8635	−1960	124	78	21
0.06	0.060 0090	10 0053	15	1.001 2036	4364	679	0.06	1.062 3835	9 6785	−1850	202	101	23
0.07	0.070 0143	10 0071	18	1.001 6400	5048	684	0.07	1.072 0620	9 5036	−1749	303	126	25
0.08	0.080 0214	10 0090	19	1.002 1448	5736	688	0.08	1.081 5656	9 3377	−1659	429	150	24
0.09	0.090 0304	10 0114	24	1.002 7184	6430	694	0.09	1.090 9033	9 1802	−1575	579	175	25
0.10	0.100 0418	10 0139	25	1.003 3614	7131	701	0.10	1.100 0835	9 0305	−1497	754	200	25
0.11	0.110 0557	10 0166	27	1.004 0745	7839	708	0.11	1.109 1140	8 8876	−1429	954	225	25
0.12	0.120 0723	10 0197	31	1.004 8584	8555	716	0.12	1.118 0016	8 7516	−1360	1179	250	25
0.13	0.130 0920	10 0230	33	1.005 7139	9281	726	0.13	1.126 7532	8 6213	−1303	1429	275	25
0.14	0.140 1150	10 0266	36	1.006 6420	1 0016	735	0.14	1.135 3745	8 4969	−1244	1704	299	24
0.15	0.150 1416	10 0304	38	1.007 6436	1 0762	746	0.15	1.143 8714	8 3776	−1193	2003	322	23
0.16	0.160 1720	10 0346	42	1.008 7198	1 1520	758	0.16	1.152 2490	8 2632	−1144	2325	345	23
0.17	0.170 2066	10 0389	43	1.009 8718	1 2289	769	0.17	1.160 5122	8 1533	−1099	2670	368	23
0.18	0.180 2455	10 0436	47	1.011 1007	1 3073	784	0.18	1.168 6655	8 0477	−1056	3038	390	22
0.19	0.190 2891	10 0485	49	1.012 4080	1 3871	798	0.19	1.176 7132	7 9461	−1016	3428	411	21
0.20	0.200 3376	10 0537	52	1.013 7951	1 4684	813	0.20	1.184 6593	7 8482	− 979	3839	432	21
0.21	0.210 3913	10 0592	55	1.015 2635	1 5514	830	0.21	1.192 5075	7 7538	− 944	4271	452	20
0.22	0.220 4505	10 0651	59	1.016 8149	1 6363	849	0.22	1.200 2613	7 6628	− 910	4723	472	20
0.23	0.230 5156	10 0711	60	1.018 4512	1 7229	866	0.23	1.207 9241	7 5749	− 879	5195	491	19
0.24	0.240 5867	10 0775	64	1.020 1741	1 8118	889	0.24	1.215 4990	7 4900	− 849	5686	509	18
0.25	0.250 6642	10 0841	66	1.021 9859	1 9026	908	0.25	1.222 9890	7 4077	− 823	6195	527	18
0.26	0.260 7483	10 0912	71	1.023 8885	1 9959	933	0.26	1.230 3967	7 3282	− 795	6722	544	17
0.27	0.270 8395	10 0985	73	1.025 8844	2 0917	958	0.27	1.237 7249	7 2511	− 771	7266	561	17
0.28	0.280 9380	10 1061	76	1.027 9761	2 1900	983	0.28	1.244 9760	7 1766	− 745	7827	577	16
0.29	0.291 0441	10 1140	79	1.030 1661	2 2913	1013	0.29	1.252 1526	7 1040	− 726	8404	593	16
0.30	0.301 1581	10 1223	83	1.032 4574	2 3954	1041	0.30	1.259 2566	7 0339	− 701	8997		16
0.31	0.311 2804	10 1310	87	1.034 8528	2 5029	1075	0.31	1.266 2905	6 9655	− 684			
0.32	0.321 4114	10 1399	89	1.037 3557	2 6136	1107	0.32	1.273 2560	6 8994	− 661			
0.33	0.331 5513	10 1492	93	1.039 9693	2 7281	1145	0.33	1.280 1554	6 8348	− 646			
0.34	0.341 7005	10 1588	96	1.042 6974	2 8464	1183	0.34	1.286 9902	6 7722	− 626			
0.35	0.351 8593	10 1689	101	1.045 5438	2 9689	1225	0.35	1.293 7624	6 7112	− 610			
0.36	0.362 0282	10 1793	104	1.048 5127	3 0957	1268	0.36	1.300 4736	6 6519	− 593			
0.37	0.372 2075	10 1901	108	1.051 6084	3 2273	1316	0.37	1.307 1255	6 5940	− 579			
0.38	0.382 3976	10 2012	111	1.054 8357	3 3639	1366	0.38	1.313 7195	6 5377	− 563			
0.39	0.392 5988	10 2128	116	1.058 1996	3 5060	1421	0.39	1.320 2572	6 4828	− 549			
0.40	0.402 8116	10 2249	121	1.061 7056	3 6537	1477	0.40	1.326 7400	6 4292	− 536			
0.41	0.413 0365	10 2372	123	1.065 3593	3 8077	1540	0.41	1.333 1692	6 3769	− 523			
0.42	0.423 2737	10 2501	129	1.069 1670	3 9682	1605	0.42	1.339 5461	6 3259	− 510			
0.43	0.433 5238	10 2635	134	1.073 1352	4 1360	1678	0.43	1.345 8720	6 2762	− 497			
0.44	0.443 7873	10 2772	137	1.077 2712	4 3113	1753	0.44	1.352 1482	6 2274	− 488			
0.45	0.454 0645	10 2915	143	1.081 5825	4 4948	1835	0.45	1.358 3756	6 1799	− 475			
0.46	0.464 3560	10 3062	147	1.086 0773	4 6871	1923	0.46	1.364 5555	6 1335	− 464			
0.47	0.474 6622	10 3215	153	1.090 7644	4 8891	2020	0.47	1.370 6890	6 0879	− 456			
0.48	0.484 9837	10 3372	157	1.095 6535	5 1012	2121	0.48	1.376 7769	6 0435	− 444			
0.49	0.495 3209	10 3536	164	1.100 7547	5 3245	2233	0.49	1.382 8204	6 0002	− 433			
0.50	0.505 6745	10 3704	168	1.106 0792	5 5597	2352	0.50	1.388 8206	5 9574	− 428			
0.51	0.516 0449	10 3880	176	1.111 6389	5 8083	2486	0.51	1.394 7780	5 9158	− 416			
0.52	0.526 4329	10 4060	180	1.117 4472	6 0707	2624	0.52	1.400 6938	5 8750	− 408			
0.53	0.536 8389	10 4248	188	1.123 5179	6 3487	2780	0.53	1.406 5688	5 8350	− 400			
0.54	0.547 2637	10 4442	194	1.129 8666	6 6435	2948	0.54	1.412 4038	5 7959	− 391			
0.55	0.557 7079	10 4642	200	1.136 5101	6 9566	3131	0.55	1.418 1997	5 7574	− 385			
0.56	0.568 1721	10 4850	208	1.143 4667	7 2899	3333	0.56	1.423 9571	5 7198	− 376			
0.57	0.578 6571	10 5065	215	1.150 7566	7 6452	3553	0.57	1.429 6769	5 6828	− 370			
0.58	0.589 1636	10 5288	223	1.158 4018	8 0247	3795	0.58	1.435 3597	5 6466	− 362			
0.59	0.599 6924	10 5519	231	1.166 4265	8 4310	4063	0.59	1.441 0063	5 6112	− 354			
0.60	0.610 2443		240	1.174 8575		4331	0.60	1.446 6175		− 346			

x	ξ			Q			q	f		
0.00	0.000 0000	58	115	1.0000 0000	30 1618	3216	+0.000	3.00 0000	-7483	35
0.01	0.000 0058	173	115	1.0030 1618	30 4895	3277	0.001	2.99 2517	-7447	36
0.02	0.000 0231	292	119	1.0060 6513	30 8239	3344	0.002	2.98 5070	-7414	33
0.03	0.000 0523	413	121	1.0091 4752	31 1649	3410	0.003	2.97 7656	-7379	35
0.04	0.000 0936	535	122	1.0122 6401	31 5128	3479	0.004	2.97 0277	-7344	35
0.05	0.000 1471	660	125	1.0154 1529	31 8679	3551	+0.005	2.96 2933	-7311	33
0.06	0.000 2131	787	127	1.0186 0208	32 2304	3625	0.006	2.95 5622	-7278	33
0.07	0.000 2918	917	130	1.0218 2512	32 6004	3700	0.007	2.94 8344	-7244	34
0.08	0.000 3835	1049	132	1.0250 8516	32 9782	3778	0.008	2.94 1100	-7211	33
0.09	0.000 4884	1182	133	1.0283 8298	33 3640	3858	0.009	2.93 3889	-7178	33
0.10	0.000 6066	1320	138	1.0317 1938	33 7583	3943	+0.010	2.92 6711	-7145	33
0.11	0.000 7386	1459	139	1.0350 9521	34 1610	4027	0.011	2.91 9566	-7112	33
0.12	0.000 8845	1602	143	1.0385 1131	34 5727	4117	0.012	2.91 2454	-7081	31
0.13	0.001 0447	1746	144	1.0419 6858	34 9934	4207	0.013	2.90 5373	-7048	33
0.14	0.001 2193	1894	148	1.0454 6792	35 4236	4302	0.014	2.89 8325	-7016	32
0.15	0.001 4087	2044	150	1.0490 1028	35 8637	4401	+0.015	2.89 1309	-6985	31
0.16	0.001 6131	2199	155	1.0525 9665	36 3138	4501	0.016	2.88 4324	-6953	32
0.17	0.001 8330	2355	156	1.0562 2803	36 7743	4605	0.017	2.87 7371	-6922	31
0.18	0.002 0685	2514	159	1.0599 0546	37 2457	4714	0.018	2.87 0449	-6891	31
0.19	0.002 3199	2678	164	1.0636 3003	37 7282	4825	0.019	2.86 3558	-6860	31
0.20	0.002 5877	2845	167	1.0674 0285	38 2224	4942	+0.020	2.85 6698	-6829	31
0.21	0.002 8722	3014	169	1.0712 2509	38 7285	5061	0.021	2.84 9869	-6799	30
0.22	0.003 1736	3188	174	1.0750 9794	39 2471	5186	0.022	2.84 3070	-6768	31
0.23	0.003 4924	3365	177	1.0790 2265	39 7787	5316	0.023	2.83 6302	-6739	29
0.24	0.003 8289	3546	181	1.0830 0052	40 3236	5449	0.024	2.82 9563	-6709	30
0.25	0.004 1835	3731	185	1.0870 3288	40 8823	5587	+0.025	2.82 2854	-6679	30
0.26	0.004 5566	3919	188	1.0911 2111	41 4556	5733	0.026	2.81 6175	-6649	30
0.27	0.004 9485	4113	194	1.0952 6667	42 0438	5882	0.027	2.80 9526	-6620	29
0.28	0.005 3598	4310	197	1.0994 7105	42 6475	6037	0.028	2.80 2906	-6591	29
0.29	0.005 7908	4513	203	1.1037 3580	43 2675	6200	0.029	2.79 6315	-6562	29
0.30	0.006 2421		203	1.1080 6255		6368	+0.030	2.78 9753		29

Hyperbola

x	ξ			Q			q	f		
0.00	0.000 0000	57	115	1.0000 0000	-29 8403	3216	-0.000	3.00 0000	7518	35
0.01	0.000 0057	169	112	0.9970 1597	-29 5251	3151	0.001	3.00 7518	7552	34
0.02	0.000 0226	280	111	0.9940 6346	-29 2157	3094	0.002	3.01 5070	7589	37
0.03	0.000 0506	388	108	0.9911 4189	-28 9123	3034	0.003	3.02 2659	7624	35
0.04	0.000 0894	495	107	0.9882 5066	-28 6145	2978	0.004	3.03 0283	7659	35
0.05	0.000 1389	599	104	0.9853 8921	-28 3221	2924	-0.005	3.03 7942	7697	38
0.06	0.000 1988	703	104	0.9825 5700	-28 0353	2868	0.006	3.04 5639	7732	35
0.07	0.000 2691	805	102	0.9797 5347	-27 7535	2818	0.007	3.05 3371	7770	38
0.08	0.000 3496	905	100	0.9769 7812	-27 4769	2766	0.008	3.06 1141	7806	36
0.09	0.000 4401	1002	97	0.9742 3043	-27 2052	2717	0.009	3.06 8947	7843	37
0.10	0.000 5403	1100	98	0.9715 0991	-26 9384	2668	-0.010	3.07 6790	7881	38
0.11	0.000 6503	1195	95	0.9688 1607	-26 6761	2623	0.011	3.08 4671	7919	38
0.12	0.000 7698	1288	93	0.9661 4846	-26 4186	2575	0.012	3.09 2590	7957	38
0.13	0.000 8986	1380	92	0.9635 0660	-26 1653	2533	0.013	3.10 0547	7994	37
0.14	0.001 0366	1472	92	0.9608 9007	-25 9166	2487	0.014	3.10 8541	8034	40
0.15	0.001 1838	1560	88	0.9582 9841	-25 6720	2446	-0.015	3.11 6575	8072	38
0.16	0.001 3398	1649	89	0.9557 3121	-25 4316	2404	0.016	3.12 4647	8111	39
0.17	0.001 5047	1735	86	0.9531 8805	-25 1952	2364	0.017	3.13 2758	8151	40
0.18	0.001 6782	1820	85	0.9506 6853	-24 9626	2326	0.018	3.14 0909	8190	39
0.19	0.001 8602	1905	85	0.9481 7227	-24 7341	2285	0.019	3.14 9099	8230	40
0.20	0.002 0507	1987	82	0.9456 9886	-24 5091	2250	-0.020	3.15 7329	8271	41
0.21	0.002 2494	2068	81	0.9432 4795	-24 2879	2212	0.021	3.16 5600	8311	40
0.22	0.002 4562	2149	81	0.9408 1916	-24 0702	2177	0.022	3.17 3911	8351	40
0.23	0.002 6711	2228	79	0.9384 1214	-23 8559	2143	0.023	3.18 2262	8393	42
0.24	0.002 8939	2306	78	0.9360 2655	-23 6452	2107	0.024	3.19 0655	8434	41
0.25	0.003 1245	2383	77	0.9336 6203	-23 4378	2074	-0.025	3.19 9089	8475	41
0.26	0.003 3628	2459	76	0.9313 1825	-23 2334	2044	0.026	3.20 7564	8517	42
0.27	0.003 6087	2533	74	0.9289 9491	-23 0324	2010	0.027	3.21 6081	8560	43
0.28	0.003 8620	2607	74	0.9266 9167	-22 8345	1979	0.028	3.22 4641	8602	42
0.29	0.004 1227	2679	72	0.9244 0822	-22 6396	1949	0.029	3.23 3243	8645	43
0.30	0.004 3906		72	0.9221 4426		1919	-0.030	3.24 1888		43

Table V

A	Ellipse B		Ellipse C		Ellipse D		Hyperbola B		Hyperbola C		Hyperbola D	
0.000	1.00000000	1	1.000 0000	4002	1.000 0000	9998	1.00000000	1	1.000 0000	3998	1.000 0000	10002
0.001	0.99999998	5	1.000 4002	4007	0.999 0002	9994	0.99999998	5	0.999 6002	3993	1.001 0002	10006
0.002	99993	8	1.000 8009	4011	0.998 0008	9990	99993	8	0.999 2009	3989	1.002 0008	10010
0.003	99985	12	1.001 2020	4015	0.997 0018	9986	99985	12	0.998 8020	3985	1.003 0018	10014
0.004	99973	16	1.001 6035	4019	0.996 0032	9982	99973	16	0.998 4035	3981	1.004 0032	10018
0.005	99957	19	1.002 0054	4024	0.995 0050	9978	99957	19	0.998 0054	3976	1.005 0050	10022
0.006	99938	22	1.002 4078	4029	0.994 0072	9974	99938	22	0.997 6078	3972	1.006 0072	10026
0.007	99916	26	1.002 8107	4033	0.993 0098	9970	99916	26	0.997 2106	3968	1.007 0098	10030
0.008	99890	29	1.003 2140	4037	0.992 0128	9966	99890	29	0.996 8138	3963	1.008 0128	10034
0.009	99861	33	1.003 6177	4041	0.991 0162	9962	99861	32	0.996 4175	3959	1.009 0162	10038
0.010	0.99999828	36	1.004 0218	4046	0.990 0200	9958	0.99999829	36	0.996 0216	3955	1.010 0200	10042
0.011	99792	40	1.004 4264	4051	0.989 0242	9954	99793	39	0.995 6261	3950	1.011 0242	10046
0.012	99752	43	1.004 8315	4055	0.988 0288	9950	99754	43	0.995 2311	3947	1.012 0288	10050
0.013	99709	46	1.005 2370	4059	0.987 0338	9946	99711	46	0.994 8364	3942	1.013 0338	10054
0.014	99663	50	1.005 6429	4064	0.986 0392	9942	99665	49	0.994 4422	3938	1.014 0392	10058
0.015	99613	53	1.006 0493	4068	0.985 0450	9938	99616	53	0.994 0484	3933	1.015 0450	10062
0.016	99560	57	1.006 4561	4073	0.984 0512	9934	99563	57	0.993 6551	3930	1.016 0512	10066
0.017	99503	61	1.006 8634	4077	0.983 0578	9930	99506	59	0.993 2621	3925	1.017 0578	10070
0.018	99442	63	1.007 2711	4082	0.982 0648	9926	99447	63	0.992 8696	3921	1.018 0648	10074
0.019	99379	68	1.007 6793	4086	0.981 0722	9922	99384	67	0.992 4775	3916	1.019 0722	10078
0.020	0.99999311	71	1.008 0879	4090	0.980 0800	9917	0.99999317	70	0.992 0859	3913	1.020 0800	10081
0.021	99240	74	1.008 4969	4095	0.979 0883	9914	99247	73	0.991 6946	3908	1.021 0881	10086
0.022	99166	78	1.008 9064	4100	0.978 0969	9910	99174	76	0.991 3038	3904	1.022 0967	10090
0.023	99088	81	1.009 3164	4104	0.977 1059	9906	99098	80	0.990 9134	3900	1.023 1057	10094
0.024	99007	84	1.009 7268	4109	0.976 1153	9902	99018	84	0.990 5234	3896	1.024 1151	10098
0.025	98923	89	1.010 1377	4113	0.975 1251	9898	98934	86	0.990 1338	3892	1.025 1249	10102
0.026	98834	91	1.010 5490	4118	0.974 1353	9894	98848	90	0.989 7446	3887	1.026 1351	10106
0.027	98743	95	1.010 9608	4122	0.973 1459	9890	98758	94	0.989 3559	3884	1.027 1457	10110
0.028	98648	99	1.011 3730	4127	0.972 1569	9886	98664	97	0.988 9675	3879	1.028 1567	10114
0.029	98549	102	1.011 7857	4132	0.971 1683	9881	98567	100	0.988 5796	3875	1.029 1681	10117
0.030	0.99998447	106	1.012 1989	4136	0.970 1802	9878	0.99998467	103	0.988 1921	3871	1.030 1798	10122
0.031	98341	109	1.012 6125	4140	0.969 1924	9874	98364	107	0.987 8050	3867	1.031 1920	10126
0.032	98232	113	1.013 0265	4145	0.968 2050	9870	98257	110	0.987 4183	3862	1.032 2046	10130
0.033	98119	116	1.013 4410	4150	0.967 2180	9866	98147	114	0.987 0321	3859	1.033 2176	10134
0.034	98003	119	1.013 8560	4155	0.966 2314	9861	98033	117	0.986 6462	3854	1.034 2310	10138
0.035	97884	124	1.014 2715	4159	0.965 2453	9858	97916	120	0.986 2608	3851	1.035 2448	10141
0.036	97760	126	1.014 6874	4163	0.964 2595	9854	97796	124	0.985 8757	3846	1.036 2589	10146
0.037	97634	131	1.015 1037	4169	0.963 2741	9850	97672	127	0.985 4911	3843	1.037 2735	10150
0.038	97503	133	1.015 5206	4173	0.962 2891	9845	97545	130	0.985 1068	3838	1.038 2885	10154
0.039	97370	138	1.015 9379	4177	0.961 3046	9842	97415	134	0.984 7230	3834	1.039 3039	10157
0.040	0.99997232	140	1.016 3556	4182	0.960 3204	9838	0.99997281	137	0.984 3396	3830	1.040 3196	10162
0.041	97092	145	1.016 7738	4187	0.959 3366	9834	97144	140	0.983 9566	3826	1.041 3358	10166
0.042	96947	147	1.017 1925	4192	0.958 3532	9829	97004	144	0.983 5740	3822	1.042 3524	10170
0.043	96800	152	1.017 6117	4196	0.957 3703	9826	96860	147	0.983 1918	3818	1.043 3694	10173
0.044	96648	155	1.018 0313	4201	0.956 3877	9822	96713	150	0.982 8100	3814	1.044 3867	10178
0.045	96493	158	1.018 4514	4206	0.955 4055	9817	96563	154	0.982 4286	3810	1.045 4045	10182
0.046	96335	162	1.018 8720	4210	0.954 4238	9814	96409	157	0.982 0476	3806	1.046 4227	10185
0.047	96173	165	1.019 2930	4215	0.953 4424	9809	96252	160	0.981 6670	3801	1.047 4412	10190
0.048	96008	169	1.019 7145	4220	0.952 4615	9806	96092	164	0.981 2869	3798	1.048 4602	10194
0.049	95839	173	1.020 1365	4224	0.951 4809	9802	95928	167	0.980 9071	3794	1.049 4796	10197
0.050	0.99995666	176	1.020 5589	4230	0.950 5007	9797	0.99995761	170	0.980 5277	3790	1.050 4993	10202
0.051	95490	180	1.020 9819	4234	0.949 5210	9794	95591	174	0.980 1487	3786	1.051 5195	10205
0.052	95310	183	1.021 4053	4239	0.948 5416	9789	95417	177	0.979 7701	3782	1.052 5400	10210
0.053	95127	187	1.021 8292	4243	0.947 5627	9786	95240	180	0.979 3919	3778	1.053 5610	10213
0.054	94940	190	1.022 2535	4248	0.946 5841	9781	95060	183	0.979 0141	3774	1.054 5823	10218
0.055	94750	194	1.022 6783	4254	0.945 6060	9777	94877	187	0.978 6367	3770	1.055 6041	10221
0.056	94556	197	1.023 1037	4257	0.944 6283	9774	94690	190	0.978 2597	3766	1.056 6262	10226
0.057	94359	201	1.023 5294	4263	0.943 6509	9769	94500	194	0.977 8831	3762	1.057 6488	10229
0.058	94158	205	1.023 9557	4268	0.942 6740	9766	94306	196	0.977 5069	3758	1.058 6717	10234
0.059	93953	208	1.024 3825	4272	0.941 6974	9761	94110	201	0.977 1311	3755	1.059 6951	10237
0.060	0.99993745	212	1.024 8097	4277	0.940 7213	9757	0.99993909	203	0.976 7556	3750	1.060 7188	10242
0.061	93533	215	1.025 2374	4282	0.939 7456	9754	93706	206	0.976 3806	3746	1.061 7430	10245
0.062	93318	219	1.025 6656	4287	0.938 7702	9749	93500	210	0.976 0060	3743	1.062 7675	10249
0.063	93099	222	1.026 0943	4292	0.937 7953	9745	93290	214	0.975 6317	3739	1.063 7924	10254
0.064	92877	226	1.026 5235	4296	0.936 8208	9741	93076	216	0.975 2578	3734	1.064 8178	10257
0.065	92651	230	1.026 9531	4301	0.935 8467	9738	92860	220	0.974 8844	3731	1.065 8435	10261
0.066	92421	233	1.027 3832	4307	0.934 8729	9733	92640	223	0.974 5113	3727	1.066 8696	10266
0.067	92188	237	1.027 8139	4311	0.933 8996	9729	92417	226	0.974 1386	3723	1.067 8962	10269
0.068	91951	240	1.028 2450	4316	0.932 9267	9725	92191	230	0.973 7663	3719	1.068 9231	10273
0.069	91711	244	1.028 6766	4321	0.931 9542	9721	91961	233	0.973 3944	3716	1.069 9504	10278
0.070	0.99991467	248	1.029 1087	4326	0.930 9821	9717	0.99991728	236	0.973 0228	3711	1.070 9782	10281
0.071	91219	251	1.029 5413	4331	0.930 0104	9713	91492	239	0.972 6517	3708	1.072 0063	10285
0.072	90968	255	1.029 9744	4335	0.929 0391	9709	91253	243	0.972 2809	3704	1.073 0348	10289
0.073	90713	258	1.030 4079	4341	0.928 0682	9705	91010	246	0.971 9105	3700	1.074 0637	10293
0.074	90455	262	1.030 8420	4346	0.927 0977	9701	90764	249	0.971 5405	3696	1.075 0930	10297
0.075	0.99990193		1.031 2766		0.926 1276		0.99990515		0.971 1709		1.076 1227	

Table V (cont'd)

	Ellipse						Hyperbola					
A	B	Δ	C	Δ	D	Δ	B	Δ	C	Δ	D	Δ
0.075	0.99990193	266	1.031 2766	4350	0.926 1276	9697	0.99990515	253	0.971 1709	3692	1.076 1227	10302
0.076	89927	269	1.031 7116	4356	0.925 1579	9693	90262	256	0.970 8017	3688	1.077 1529	10305
0.077	89658	273	1.032 1472	4360	0.924 1886	9689	90006	259	0.970 4329	3685	1.078 1834	10309
0.078	89385	276	1.032 5832	4366	0.923 2197	9685	89747	262	0.970 0644	3681	1.079 2143	10313
0.079	89109	280	1.033 0198	4370	0.922 2512	9681	89485	266	0.969 6963	3677	1.080 2456	10317
0.080	0.99988829	284	1.033 4568	4376	0.921 2831	9676	0.99989219	269	0.969 3286	3673	1.081 2773	10321
0.081	88545	287	1.033 8944	4381	0.920 3155	9673	88950	272	0.968 9613	3670	1.082 3094	10325
0.082	88258	291	1.034 3325	4385	0.919 3482	9669	88678	275	0.968 5943	3665	1.083 3419	10329
0.083	87967	294	1.034 7710	4391	0.918 3813	9665	88403	279	0.968 2278	3662	1.084 3748	10333
0.084	87673	299	1.035 2101	4395	0.917 4148	9660	88124	281	0.967 8616	3658	1.085 4081	10336
0.085	87374	301	1.035 6496	4401	0.916 4488	9657	87843	285	0.967 4958	3655	1.086 4417	10341
0.086	87073	306	1.036 0897	4406	0.915 4831	9652	87558	289	0.967 1303	3650	1.087 4758	10345
0.087	86767	309	1.036 5303	4410	0.914 5179	9649	87269	291	0.966 7653	3647	1.088 5103	10349
0.088	86458	313	1.036 9713	4416	0.913 5530	9645	86978	295	0.966 4006	3643	1.089 5452	10353
0.089	86145	316	1.037 4129	4421	0.912 5885	9640	86683	298	0.966 0363	3640	1.090 5805	10357
0.090	0.99985829	320	1.037 8550	4426	0.911 6245	9636	0.99986385	301	0.965 6723	3635	1.091 6162	10360
0.091	85509	324	1.038 2976	4432	0.910 6609	9633	86084	305	0.965 3088	3632	1.092 6522	10365
0.092	85185	327	1.038 7408	4436	0.909 6976	9628	85779	307	0.964 9456	3628	1.093 6887	10369
0.093	84858	331	1.039 1844	4441	0.908 7348	9625	85472	311	0.964 5828	3625	1.094 7256	10372
0.094	84527	334	1.039 6285	4447	0.907 7723	9620	85161	315	0.964 2203	3621	1.095 7628	10377
0.095	84193	339	1.040 0732	4451	0.906 8103	9616	84846	317	0.963 8582	3617	1.096 8005	10381
0.096	83854	341	1.040 5183	4457	0.905 8487	9612	84529	321	0.963 4965	3613	1.097 8386	10384
0.097	83513	346	1.040 9640	4462	0.904 8875	9609	84208	323	0.963 1352	3610	1.098 8770	10389
0.098	83167	349	1.041 4102	4467	0.903 9266	9604	83885	327	0.962 7742	3606	1.099 9159	10392
0.099	82818	353	1.041 8569	4473	0.902 9662	9600	83558	331	0.962 4136	3602	1.100 9551	10397
0.100	0.99982465	357	1.042 3042	4477	0.902 0062	9596	0.99983227	333	0.962 0534	3599	1.101 9948	10400
0.101	82108	360	1.042 7519	4483	0.901 0466	9592	82894	337	0.961 6935	3595	1.103 0348	10405
0.102	81748	364	1.043 2002	4488	0.900 0874	9588	82557	340	0.961 3340	3591	1.104 0753	10408
0.103	81384	368	1.043 6490	4493	0.899 1286	9584	82217	343	0.960 9749	3587	1.105 1161	10412
0.104	81016	371	1.044 0983	4499	0.898 1702	9580	81874	346	0.960 6162	3584	1.106 1573	10417
0.105	80645	375	1.044 5482	4503	0.897 2122	9576	81528	349	0.960 2578	3581	1.107 1990	10420
0.106	80270	379	1.044 9985	4509	0.896 2546	9571	81179	353	0.959 8997	3577	1.108 2410	10424
0.107	79891	382	1.045 4494	4514	0.895 2975	9568	80826	356	0.959 5420	3573	1.109 2834	10429
0.108	79509	386	1.045 9008	4520	0.894 3407	9564	80470	359	0.959 1847	3569	1.110 3263	10432
0.109	79123	390	1.046 3528	4524	0.893 3843	9559	80111	362	0.958 8278	3566	1.111 3695	10436
0.110	0.99978733	393	1.046 8052	4530	0.892 4284	9556	0.99979749	366	0.958 4712	3562	1.112 4131	10440
0.111	78340	397	1.047 2582	4536	0.891 4728	9552	79383	369	0.958 1150	3559	1.113 4571	10444
0.112	77943	401	1.047 7118	4540	0.890 5176	9547	79014	371	0.957 7591	3555	1.114 5015	10448
0.113	77542	405	1.048 1658	4546	0.889 5629	9544	78643	375	0.957 4036	3551	1.115 5463	10452
0.114	77137	408	1.048 6204	4551	0.888 6085	9539	78268	379	0.957 0485	3548	1.116 5915	10456
0.115	76729	412	1.049 0755	4557	0.887 6546	9536	77889	381	0.956 6937	3544	1.117 6371	10461
0.116	76317	416	1.049 5312	4562	0.886 7010	9531	77508	385	0.956 3393	3541	1.118 6832	10463
0.117	75901	419	1.049 9874	4567	0.885 7479	9527	77123	387	0.955 9852	3537	1.119 7295	10468
0.118	75482	423	1.050 4441	4573	0.884 7952	9523	76736	391	0.955 6315	3534	1.120 7763	10472
0.119	75059	427	1.050 9014	4578	0.883 8429	9520	76345	394	0.955 2781	3530	1.121 8235	10476
0.120	0.99974632	430	1.051 3592	4583	0.882 8909	9515	0.99975951	398	0.954 9251	3526	1.122 8711	10480
0.121	74202	435	1.051 8175	4589	0.881 9394	9511	75553	400	0.954 5725	3523	1.123 9191	10484
0.122	73767	438	1.052 2764	4594	0.880 9883	9507	75153	404	0.954 2202	3519	1.124 9675	10488
0.123	73329	441	1.052 7358	4600	0.880 0376	9503	74749	407	0.953 8683	3516	1.126 0163	10491
0.124	72888	446	1.053 1958	4605	0.879 0873	9499	74342	410	0.953 5167	3512	1.127 0654	10496
0.125	72442	449	1.053 6563	4610	0.878 1374	9495	73932	413	0.953 1655	3509	1.128 1150	10500
0.126	71993	453	1.054 1173	4616	0.877 1879	9491	73519	416	0.952 8146	3505	1.129 1650	10503
0.127	71540	457	1.054 5789	4621	0.876 2388	9486	73103	419	0.952 4641	3502	1.130 2153	10508
0.128	71083	460	1.055 0410	4627	0.875 2902	9483	72684	423	0.952 1139	3498	1.131 2661	10512
0.129	70623	464	1.055 5037	4632	0.874 3419	9479	72261	426	0.951 7641	3494	1.132 3173	10515
0.130	0.99970159	468	1.055 9669	4638	0.873 3940	9474	0.99971835	429	0.951 4147	3492	1.133 3688	10520
0.131	69691	472	1.056 4307	4644	0.872 4466	9471	71406	432	0.951 0655	3487	1.134 4208	10523
0.132	69219	475	1.056 8951	4648	0.871 4995	9466	70974	435	0.950 7168	3484	1.135 4731	10528
0.133	68744	480	1.057 3599	4655	0.870 5529	9463	70539	438	0.950 3684	3481	1.136 5259	10531
0.134	68264	483	1.057 8254	4660	0.869 6066	9458	70101	442	0.950 0203	3477	1.137 5790	10535
0.135	67781	486	1.058 2914	4665	0.868 6608	9455	69659	444	0.949 6726	3474	1.138 6325	10540
0.136	67295	491	1.058 7579	4671	0.867 7153	9450	69215	448	0.949 3252	3470	1.139 6865	10543
0.137	66804	494	1.059 2250	4676	0.866 7703	9446	68767	451	0.948 9782	3467	1.140 7408	10547
0.138	66310	498	1.059 6926	4683	0.865 8257	9442	68316	454	0.948 6315	3463	1.141 7955	10551
0.139	65812	502	1.060 1609	4687	0.864 8815	9438	67862	457	0.948 2852	3460	1.142 8506	10556
0.140	0.99965310	505	1.060 6296	4693	0.863 9377	9434	0.99967405	461	0.947 9392	3457	1.143 9062	10559
0.141	64805	510	1.061 0989	4699	0.862 9943	9430	66944	463	0.947 5935	3453	1.144 9621	10563
0.142	64295	513	1.061 5688	4705	0.862 0513	9426	66481	467	0.947 2482	3449	1.146 0184	10567
0.143	63782	517	1.062 0393	4710	0.861 1087	9422	66014	469	0.946 9033	3447	1.147 0751	10571
0.144	63265	521	1.062 5103	4716	0.860 1665	9418	65545	473	0.946 5586	3442	1.148 1322	10575
0.145	62744	524	1.062 9819	4721	0.859 2247	9414	65072	476	0.946 2144	3440	1.149 1897	10579
0.146	62220	528	1.063 4540	4727	0.858 2833	9409	64596	479	0.945 8704	3435	1.150 2476	10583
0.147	61692	532	1.063 9267	4733	0.857 3424	9406	64117	482	0.945 5269	3433	1.151 3059	10587
0.148	61160	536	1.064 4000	4738	0.856 4018	9402	63635	486	0.945 1836	3429	1.152 3646	10590
0.149	60624	540	1.064 8738	4744	0.855 4616	9397	63149	488	0.944 8407	3426	1.153 4236	10595
0.150	0.99960084		1.065 3482		0.854 5219		0.99962661		0.944 4981		1.154 4831	

Table V (cont'd)

	Ellipse						Hyperbola					
A	B		C		D		B		C		D	
0.150	0.99960084	543	1.065 3482	4750	0.854 5219	9393	0.99962661	492	0.944 4981	3422	1.154 4831	10599
0.151	59541	548	1.065 8232	4755	0.853 5826	9390	62169	494	0.944 1559	3419	1.155 5430	10603
0.152	58993	551	1.066 2987	4762	0.852 6436	9385	61675	498	0.943 8140	3416	1.156 6033	10606
0.153	58442	555	1.066 7749	4767	0.851 7051	9381	61177	501	0.943 4724	3412	1.157 6639	10611
0.154	57887	559	1.067 2516	4772	0.850 7670	9377	60676	504	0.943 1312	3408	1.158 7250	10615
0.155	57328	562	1.067 7288	4779	0.849 8293	9373	60172	507	0.942 7904	3406	1.159 7865	10618
0.156	56766	566	1.068 2067	4784	0.848 8920	9369	59665	510	0.942 4498	3402	1.160 8483	10623
0.157	56200	571	1.068 6851	4790	0.847 9551	9365	59155	513	0.942 1096	3399	1.161 9106	10626
0.158	55629	574	1.069 1641	4796	0.847 0186	9361	58642	517	0.941 7697	3395	1.162 9732	10631
0.159	55055	578	1.069 6437	4801	0.846 0825	9357	58125	519	0.941 4302	3392	1.164 0363	10634
0.160	0.99954477	581	1.070 1238	4807	0.845 1468	9353	0.99957606	523	0.941 0910	3389	1.165 0997	10638
0.161	53896	586	1.070 6045	4814	0.844 2115	9348	57083	525	0.940 7521	3385	1.166 1635	10643
0.162	53310	589	1.071 0859	4819	0.843 2767	9345	56558	529	0.940 4136	3382	1.167 2278	10646
0.163	52721	593	1.071 5678	4824	0.842 3422	9340	56029	532	0.940 0754	3379	1.168 2924	10650
0.164	52128	597	1.072 0502	4831	0.841 4082	9337	55497	535	0.939 7375	3375	1.169 3574	10654
0.165	51531	601	1.072 5333	4837	0.840 4745	9332	54962	538	0.939 4000	3372	1.170 4228	10659
0.166	50930	605	1.073 0170	4842	0.839 5413	9329	54424	541	0.939 0628	3369	1.171 4887	10662
0.167	50325	609	1.073 5012	4849	0.838 6084	9324	53883	544	0.938 7259	3366	1.172 5549	10666
0.168	49716	612	1.073 9861	4854	0.837 6760	9320	53339	547	0.938 3893	3362	1.173 6215	10670
0.169	49104	616	1.074 4715	4860	0.836 7440	9316	52792	551	0.938 0531	3359	1.174 6885	10674
0.170	0.99948488	620	1.074 9575	4866	0.835 8124	9312	0.99952241	553	0.937 7172	3355	1.175 7559	10678
0.171	47868	624	1.075 4441	4872	0.834 8812	9308	51688	556	0.937 3817	3353	1.176 8237	10682
0.172	47244	628	1.075 9313	4878	0.833 9504	9304	51132	560	0.937 0464	3349	1.177 8919	10686
0.173	46616	632	1.076 4191	4884	0.833 0200	9299	50572	563	0.936 7115	3345	1.178 9605	10689
0.174	45984	636	1.076 9075	4890	0.832 0901	9296	50009	565	0.936 3770	3343	1.180 0294	10694
0.175	45348	639	1.077 3965	4896	0.831 1605	9292	49444	569	0.936 0427	3339	1.181 0988	10698
0.176	44709	644	1.077 8861	4902	0.830 2313	9287	48875	572	0.935 7088	3336	1.182 1686	10702
0.177	44065	647	1.078 3763	4907	0.829 3026	9283	48303	575	0.935 3752	3333	1.183 2388	10705
0.178	43418	651	1.078 8670	4914	0.828 3743	9280	47728	578	0.935 0419	3329	1.184 3093	10710
0.179	42767	655	1.079 3584	4920	0.827 4463	9275	47150	581	0.934 7090	3327	1.185 3803	10713
0.180	0.99942112	659	1.079 8504	4926	0.826 5188	9271	0.99946569	584	0.934 3763	3323	1.186 4516	10718
0.181	41453	663	1.080 3430	4932	0.825 5917	9267	45985	587	0.934 0440	3320	1.187 5234	10721
0.182	40790	666	1.080 8362	4938	0.824 6650	9263	45398	590	0.933 7120	3316	1.188 5955	10726
0.183	40124	671	1.081 3300	4944	0.823 7387	9259	44808	593	0.933 3804	3313	1.189 6681	10729
0.184	39453	674	1.081 8244	4950	0.822 8128	9255	44215	596	0.933 0491	3311	1.190 7410	10734
0.185	38779	679	1.082 3194	4957	0.821 8873	9251	43619	600	0.932 7180	3306	1.191 8144	10737
0.186	38100	682	1.082 8151	4962	0.820 9622	9247	43019	602	0.932 3874	3304	1.192 8881	10741
0.187	37418	686	1.083 3113	4969	0.820 0375	9242	42417	605	0.932 0570	3301	1.193 9622	10745
0.188	36732	690	1.083 8082	4974	0.819 1133	9239	41812	609	0.931 7269	3297	1.195 0367	10749
0.189	36042	694	1.084 3056	4981	0.818 1894	9234	41203	611	0.931 3972	3294	1.196 1116	10754
0.190	0.99935348	698	1.084 8037	4987	0.817 2660	9230	0.99940592	615	0.931 0678	3291	1.197 1870	10757
0.191	34650	702	1.085 3024	4993	0.816 3430	9227	39977	617	0.930 7387	3288	1.198 2627	10761
0.192	33948	706	1.085 8017	5000	0.815 4203	9222	39360	621	0.930 4099	3284	1.199 3388	10765
0.193	33242	710	1.086 3017	5005	0.814 4981	9218	38739	623	0.930 0815	3282	1.200 4153	10769
0.194	32532	713	1.086 8022	5012	0.813 5763	9214	38116	627	0.929 7533	3278	1.201 4922	10772
0.195	31819	718	1.087 3034	5018	0.812 6549	9210	37489	630	0.929 4255	3275	1.202 5694	10777
0.196	31101	721	1.087 8052	5024	0.811 7339	9205	36859	632	0.929 0980	3272	1.203 6471	10781
0.197	30380	726	1.088 3076	5030	0.810 8134	9202	36227	636	0.928 7708	3269	1.204 7252	10785
0.198	29654	729	1.088 8106	5037	0.809 8932	9198	35591	639	0.928 4439	3266	1.205 8037	10788
0.199	28925	733	1.089 3143	5043	0.808 9734	9193	34952	642	0.928 1173	3262	1.206 8825	10793
0.200	0.99928192	738	1.089 8186	5049	0.808 0541	9189	0.99934310	644	0.927 7911	3259	1.207 9618	10797
0.201	27454	741	1.090 3235	5055	0.807 1352	9186	33666	648	0.927 4652	3257	1.209 0415	10800
0.202	26713	745	1.090 8290	5062	0.806 2166	9181	33018	651	0.927 1395	3253	1.210 1215	10805
0.203	25968	749	1.091 3352	5068	0.805 2985	9177	32367	654	0.926 8142	3250	1.211 2020	10808
0.204	25219	753	1.091 8420	5075	0.804 3808	9173	31713	657	0.926 4892	3247	1.212 2828	10813
0.205	24466	757	1.092 3495	5081	0.803 4635	9169	31056	659	0.926 1645	3243	1.213 3641	10816
0.206	23709	761	1.092 8576	5087	0.802 5466	9165	30397	663	0.925 8402	3241	1.214 4457	10820
0.207	22948	765	1.093 3663	5093	0.801 6301	9160	29734	666	0.925 5161	3238	1.215 5277	10825
0.208	22183	769	1.093 8756	5100	0.800 7141	9157	29068	669	0.925 1923	3234	1.216 6102	10828
0.209	21414	773	1.094 3856	5106	0.799 7984	9153	28399	672	0.924 8689	3231	1.217 6930	10832
0.210	0.99920641	777	1.094 8962	5113	0.798 8831	9148	0.99927727	675	0.924 5458	3229	1.218 7762	10836
0.211	19864	781	1.095 4075	5119	0.797 9683	9144	27052	678	0.924 2229	3225	1.219 8598	10840
0.212	19083	785	1.095 9194	5126	0.797 0539	9141	26374	680	0.923 9004	3222	1.220 9438	10844
0.213	18298	788	1.096 4320	5132	0.796 1398	9136	25694	684	0.923 5782	3219	1.222 0282	10848
0.214	17510	793	1.096 9452	5138	0.795 2262	9132	25010	687	0.923 2563	3216	1.223 1130	10852
0.215	16717	797	1.097 4590	5145	0.794 3130	9128	24323	690	0.922 9347	3213	1.224 1982	10856
0.216	15920	801	1.097 9735	5151	0.793 4002	9123	23633	693	0.922 6134	3210	1.225 2838	10860
0.217	15119	804	1.098 4886	5158	0.792 4879	9120	22940	696	0.922 2924	3206	1.226 3698	10864
0.218	14315	809	1.099 0044	5165	0.791 5759	9116	22244	698	0.921 9718	3204	1.227 4562	10868
0.219	13506	813	1.099 5209	5171	0.790 6643	9111	21546	702	0.921 6514	3201	1.228 5430	10871
0.220	0.99912693	817	1.100 0380	5177	0.789 7532	9107	0.99920844	705	0.921 3313	3197	1.229 6301	10876
0.221	11876	821	1.100 5557	5184	0.788 8425	9104	20139	708	0.921 0116	3195	1.230 7177	10879
0.222	11055	824	1.101 0741	5191	0.787 9321	9099	19431	710	0.920 6921	3191	1.231 8056	10884
0.223	10231	829	1.101 5932	5197	0.787 0222	9095	18721	714	0.920 3730	3189	1.232 8940	10887
0.224	09402	833	1.102 1129	5203	0.786 1127	9091	18007	717	0.920 0541	3185	1.233 9827	10892
0.225	0.99908569		1.102 6332		0.785 2036		0.99917290		0.919 7356		1.235 0719	

Table V (cont'd)

A	Ellipse B		Ellipse C		Ellipse D		Hyperbola B		Hyperbola C		Hyperbola D	
0.225	0.99908569	837	1.102 6332	5211	0.785 2036	9087	0.99917290	719	0.919 7356	3183	1.235 0719	10895
0.226	07732	841	1.103 1543	5217	0.784 2949	9082	16571	723	0.919 4173	3179	1.236 1614	10900
0.227	06891	845	1.103 6760	5223	0.783 3867	9079	15848	726	0.919 0994	3177	1.237 2514	10903
0.228	06046	849	1.104 1983	5230	0.782 4788	9074	15122	728	0.918 7817	3173	1.238 3417	10907
0.229	05197	853	1.104 7213	5237	0.781 5714	9071	14394	732	0.918 4644	3171	1.239 4324	10911
0.230	0.99904344	857	1.105 2450	5244	0.780 6643	9066	0.99913662	734	0.918 1473	3167	1.240 5235	10916
0.231	03487	861	1.105 7694	5250	0.779 7577	9062	12928	738	0.917 8306	3165	1.241 6151	10919
0.232	02626	865	1.106 2944	5257	0.778 8515	9058	12190	740	0.917 5141	3161	1.242 7070	10923
0.233	01761	869	1.106 8201	5263	0.777 9457	9054	11450	744	0.917 1980	3159	1.243 7993	10927
0.234	00892	873	1.107 3464	5270	0.777 0403	9050	10706	746	0.916 8821	3155	1.244 8920	10931
0.235	00019	877	1.107 8734	5278	0.776 1353	9046	09960	749	0.916 5666	3153	1.245 9851	10935
0.236	0.99899142	881	1.108 4012	5283	0.775 2307	9041	09211	753	0.916 2513	3149	1.247 0786	10938
0.237	98261	886	1.108 9295	5291	0.774 3266	9038	08458	755	0.915 9364	3147	1.248 1724	10943
0.238	97375	889	1.109 4586	5297	0.773 4228	9033	07703	758	0.915 6217	3143	1.249 2667	10947
0.239	96486	894	1.109 9883	5304	0.772 5195	9029	06945	761	0.915 3074	3141	1.250 3614	10951
0.240	0.99895592	897	1.110 5187	5311	0.771 6166	9025	0.99906184	764	0.914 9933	3138	1.251 4565	10954
0.241	94695	902	1.111 0498	5318	0.770 7141	9021	05420	767	0.914 6795	3135	1.252 5519	10959
0.242	93793	905	1.111 5816	5324	0.769 8120	9017	04653	770	0.914 3660	3131	1.253 6478	10962
0.243	92888	910	1.112 1140	5332	0.768 9103	9013	03883	773	0.914 0529	3129	1.254 7440	10967
0.244	91978	914	1.112 6472	5338	0.768 0090	9008	03110	776	0.913 7400	3126	1.255 8407	10970
0.245	91064	918	1.113 1810	5345	0.767 1082	9005	02334	778	0.913 4274	3123	1.256 9377	10975
0.246	90146	922	1.113 7155	5352	0.766 2077	9000	01556	782	0.913 1151	3120	1.258 0352	10978
0.247	89224	926	1.114 2507	5359	0.765 3077	8996	00774	785	0.912 8031	3118	1.259 1330	10982
0.248	88298	930	1.114 7866	5366	0.764 4081	8992	0.99899989	787	0.912 4913	3114	1.260 2312	10987
0.249	87368	934	1.115 3232	5373	0.763 5089	8988	99202	791	0.912 1799	3111	1.261 3299	10990
0.250	0.99886434	938	1.115 8605	5379	0.762 6101	8984	0.99898411	793	0.911 8688	3109	1.262 4289	10994
0.251	85496	943	1.116 3984	5387	0.761 7117	8980	97618	797	0.911 5579	3105	1.263 5283	10998
0.252	84553	946	1.116 9371	5394	0.760 8137	8976	96821	799	0.911 2474	3103	1.264 6281	11002
0.253	83607	951	1.117 4765	5400	0.759 9161	8971	96022	802	0.910 9371	3100	1.265 7283	11006
0.254	82656	955	1.118 0165	5408	0.759 0190	8967	95220	805	0.910 6271	3097	1.266 8289	11010
0.255	81701	959	1.118 5573	5414	0.758 1223	8963	94415	808	0.910 3174	3094	1.267 9299	11014
0.256	80742	962	1.119 0987	5422	0.757 2260	8960	93607	811	0.910 0080	3091	1.269 0313	11017
0.257	79780	968	1.119 6409	5429	0.756 3300	8954	92796	814	0.909 6989	3088	1.270 1330	11022
0.258	78812	971	1.120 1838	5435	0.755 4346	8951	91982	817	0.909 3901	3085	1.271 2352	11026
0.259	77841	975	1.120 7273	5443	0.754 5395	8947	91165	820	0.909 0816	3083	1.272 3378	11030
0.260	0.99876866	980	1.121 2716	5450	0.753 6448	8942	0.99890345	822	0.908 7733	3079	1.273 4408	11033
0.261	75886	983	1.121 8166	5457	0.752 7506	8939	89523	826	0.908 4654	3077	1.274 5441	11038
0.262	74903	988	1.122 3623	5464	0.751 8567	8934	88697	828	0.908 1577	3074	1.275 6479	11041
0.263	73915	992	1.122 9087	5471	0.750 9633	8930	87869	832	0.907 8503	3071	1.276 7520	11046
0.264	72923	996	1.123 4558	5479	0.750 0703	8926	87037	834	0.907 5432	3068	1.277 8566	11049
0.265	71927	1000	1.124 0037	5485	0.749 1777	8922	86203	837	0.907 2364	3065	1.278 9615	11053
0.266	70927	1004	1.124 5522	5493	0.748 2855	8918	85366	840	0.906 9299	3063	1.280 0668	11058
0.267	69923	1009	1.125 1015	5500	0.747 3937	8913	84526	843	0.906 6236	3060	1.281 1726	11061
0.268	68914	1012	1.125 6515	5507	0.746 5024	8910	83683	846	0.906 3176	3056	1.282 2787	11065
0.269	67902	1017	1.126 2022	5514	0.745 6114	8905	82837	849	0.906 0120	3054	1.283 3852	11069
0.270	0.99866885	1021	1.126 7536	5522	0.744 7209	8901	0.99881988	852	0.905 7066	3052	1.284 4921	11073
0.271	65864	1025	1.127 3058	5529	0.743 8308	8897	81136	854	0.905 4014	3048	1.285 5994	11077
0.272	64839	1029	1.127 8587	5536	0.742 9411	8893	80282	858	0.905 0966	3046	1.286 7071	11081
0.273	63810	1034	1.128 4123	5543	0.742 0518	8889	79424	860	0.904 7920	3042	1.287 8152	11085
0.274	62776	1037	1.128 9666	5551	0.741 1629	8884	78564	863	0.904 4878	3040	1.288 9237	11089
0.275	61739	1042	1.129 5217	5558	0.740 2745	8881	77701	866	0.904 1838	3037	1.290 0326	11093
0.276	60697	1046	1.130 0775	5565	0.739 3864	8876	76835	869	0.903 8801	3035	1.291 1419	11096
0.277	59651	1050	1.130 6340	5573	0.738 4988	8872	75966	872	0.903 5766	3031	1.292 2515	11101
0.278	58601	1054	1.131 1913	5580	0.737 6116	8868	75094	875	0.903 2735	3029	1.293 3616	11105
0.279	57547	1059	1.131 7493	5587	0.736 7248	8864	74219	878	0.902 9706	3026	1.294 4721	11108
0.280	0.99856488	1062	1.132 3080	5595	0.735 8384	8859	0.99873341	880	0.902 6680	3023	1.295 5829	11113
0.281	55426	1067	1.132 8675	5602	0.734 9525	8856	72461	884	0.902 3657	3021	1.296 6942	11116
0.282	54359	1071	1.133 4277	5609	0.734 0669	8851	71577	886	0.902 0636	3017	1.297 8058	11121
0.283	53288	1076	1.133 9886	5617	0.733 1818	8848	70691	889	0.901 7619	3015	1.298 9179	11124
0.284	52212	1079	1.134 5503	5625	0.732 2970	8843	69802	892	0.901 4604	3012	1.300 0303	11128
0.285	51133	1084	1.135 1128	5632	0.731 4127	8838	68910	895	0.901 1592	3010	1.301 1431	11133
0.286	50049	1088	1.135 6760	5639	0.730 5289	8835	68015	898	0.900 8582	3006	1.302 2564	11136
0.287	48961	1092	1.136 2399	5647	0.729 6454	8831	67117	900	0.900 5576	3004	1.303 3700	11140
0.288	47869	1096	1.136 8046	5654	0.728 7623	8826	66217	904	0.900 2572	3001	1.304 4840	11144
0.289	46773	1101	1.137 3700	5662	0.727 8797	8823	65313	906	0.899 9571	2998	1.305 5984	11148
0.290	0.99845672	1104	1.137 9362	5670	0.726 9974	8818	0.99864407	909	0.899 6573	2996	1.306 7132	11152
0.291	44568	1109	1.138 5032	5677	0.726 1156	8814	63498	912	0.899 3577	2993	1.307 8284	11156
0.292	43459	1114	1.139 0709	5684	0.725 2342	8810	62586	915	0.899 0584	2990	1.308 9440	11160
0.293	42345	1117	1.139 6393	5692	0.724 3532	8805	61671	918	0.898 7594	2987	1.310 0600	11164
0.294	41228	1122	1.140 2085	5700	0.723 4727	8802	60753	921	0.898 4607	2985	1.311 1764	11167
0.295	40106	1126	1.140 7785	5708	0.722 5925	8797	59832	923	0.898 1622	2982	1.312 2931	11172
0.296	38980	1130	1.141 3493	5715	0.721 7128	8793	58909	927	0.897 8640	2979	1.313 4103	11176
0.297	37850	1134	1.141 9208	5722	0.720 8335	8789	57982	929	0.897 5661	2977	1.314 5279	11179
0.298	36716	1139	1.142 4930	5731	0.719 9546	8785	57053	932	0.897 2684	2973	1.315 6458	11184
0.299	35577	1143	1.143 0661	5738	0.719 0761	8781	56121	935	0.896 9711	2971	1.316 7642	11187
0.300	0.99834434		1.143 6399		0.718 1980		0.99855186		0.896 6740		1.317 8829	

Table VI

n	''E_0		''E_1		''E_0''		''E_1''		m
0,000	+0,0833333+	4161	−0,0000000−	833	−0,004167−	27	+0,000000+	15	1,000
0,001	829172	4152	000833	834	4140	26	0015	16	0,999
0,002	825020	4142	001667	833	4114	26	0031	15	0,998
0,003	820878	4131	002500	833	4088	27	0046	15	0,997
0,004	816747	4122	003333	833	4061	26	0061	15	0,996
0,005	812625	4112	004166	834	4035	27	0076	16	0,995
0,006	808513	4102	005000	833	4008	26	0092	15	0,994
0,007	804411	4092	005833	833	3982	26	0107	15	0,993
0,008	800319	4082	006666	833	3956	27	0122	15	0,992
0,009	796237	4072	007499	833	3929	26	0137	16	0,991
0,010	+0,0792165+	4062	−0,0008332−	832	−0,003903−	27	+0,000153+	15	0,990
0,011	788103	4053	009164	833	3876	26	0168	15	0,989
0,012	784050	4042	009997	833	3850	26	0183	16	0,988
0,013	780008	4033	010830	832	3824	27	0199	15	0,987
0,014	775975	4022	011662	832	3797	26	0214	15	0,986
0,015	771953	4013	012494	833	3771	26	0229	15	0,985
0,016	767940	4003	013327	831	3745	27	0244	16	0,984
0,017	763937	3993	014158	832	3718	26	0260	15	0,983
0,018	759944	3984	014990	832	3692	26	0275	15	0,982
0,019	755960	3973	015822	831	3666	27	0290	15	0,981
0,020	+0,0751987+	3964	−0,0016653−	832	−0,003639−	26	+0,000305+	16	0,980
0,021	748023	3954	017485	831	3613	26	0321	15	0,979
0,022	744069	3944	018316	830	3587	27	0336	15	0,978
0,023	740125	3935	019146	831	3560	26	0351	15	0,977
0,024	736190	3924	019977	830	3534	26	0366	16	0,976
0,025	732266	3915	020807	830	3508	26	0382	15	0,975
0,026	728351	3905	021637	830	3482	27	0397	15	0,974
0,027	724446	3896	022467	830	3455	26	0412	15	0,973
0,028	720550	3886	023297	829	3429	26	0427	15	0,972
0,029	716664	3876	024126	829	3403	27	0442	16	0,971
0,030	+0,0712788+	3866	−0,0024955−	829	−0,003376−	26	+0,000458+	15	0,970
0,031	708922	3857	025784	828	3350	26	0473	15	0,969
0,032	705065	3847	026612	828	3324	26	0488	15	0,968
0,033	701218	3837	027440	828	3298	26	0503	15	0,967
0,034	697381	3827	028268	827	3272	27	0518	16	0,966
0,035	693554	3818	029095	827	3245	26	0534	15	0,965
0,036	689736	3809	029922	827	3219	26	0549	15	0,964
0,037	685927	3798	030749	826	3193	26	0564	15	0,963
0,038	682129	3790	031575	826	3167	26	0579	15	0,962
0,039	678339	3779	032401	826	3141	26	0594	15	0,961
0,040	+0,0674560+	3770	−0,0033227−	825	−0,003115−	27	+0,000609+	15	0,960
0,041	670790	3760	034052	825	3088	26	0624	16	0,959
0,042	667030	3751	034877	824	3062	26	0640	15	0,958
0,043	663279	3741	035701	824	3036	26	0655	15	0,957
0,044	659538	3732	036525	823	3010	26	0670	15	0,956
0,045	655806	3722	037348	823	2984	26	0685	15	0,955
0,046	652084	3712	038171	823	2958	26	0700	15	0,954
0,047	648372	3703	038994	822	2932	26	0715	15	0,953
0,048	644669	3693	039816	821	2906	26	0730	15	0,952
0,049	640976	3684	040637	821	2880	26	0745	15	0,951
0,050	+0,0637292+		−0,0041458−		−0,002854−		+0,000760+		0,950

n	'E_0		'E_1		'E_0''		'E_1''		m
0,000	−0,4166667+	9995	−0,0833333+	5	+0,026389−	0	+0,015278−	0	1,000
0,001	156672	9985	833328	15	6389	1	5278	1	0,999
0,002	146687	9975	833313	25	6388	1	5277	0	0,998
0,003	136712	9965	833288	35	6387	1	5277	1	0,997
0,004	126747	9955	833253	45	6386	1	5276	0	0,996
0,005	116792	9945	833208	55	6385	2	5276	1	0,995
0,006	106847	9935	833153	65	6383	2	5275	1	0,994
0,007	096912	9925	833088	75	6381	3	5274	2	0,993
0,008	086987	9915	833013	85	6378	2	5272	1	0,992
0,009	077072	9905	832928	95	6376	4	5271	2	0,991
0,010	−0,4067167+	9895	−0,0832833+	105	+0,026372−	3	+0,015269−	1	0,990
0,011	057272	9885	832728	115	6369	4	5268	2	0,989
0,012	047387	9875	832613	125	6365	4	5266	2	0,988
0,013	037512	9865	832488	135	6361	4	5264	3	0,987
0,014	027647	9855	832353	145	6357	5	5261	2	0,986
0,015	017792	9845	832208	155	6352	5	5259	3	0,985
0,016	−0,4007947	9835	832053	165	6347	5	5256	2	0,984
0,017	−0,3998112	9825	831888	175	6342	6	5254	3	0,983
0,018	988287	9815	831713	185	6336	6	5251	3	0,982
0,019	978472	9805	831528	195	6330	6	5248	4	0,981
0,020	−0,3968667+	9795	−0,0831333+	205	+0,026324−	7	+0,015244−	3	0,980
0,021	958872	9785	831128	215	6317	7	5241	4	0,979
0,022	949087	9775	830913	225	6310	7	5237	3	0,978
0,023	939312	9765	830688	235	6303	8	5234	4	0,977
0,024	929547	9755	830453	245	6295	8	5230	4	0,976
0,025	919792	9745	830208	255	6287	8	5226	5	0,975
0,026	910047	9735	829953	265	6279	8	5221	4	0,974
0,027	900312	9725	829688	275	6271	9	5217	5	0,973
0,028	890587	9715	829413	285	6262	9	5212	4	0,972
0,029	880872	9705	829128	295	6253	10	5208	5	0,971
0,030	−0,3871167+	9695	−0,0828833+	305	+0,026243−	9	+0,015203−	5	0,970
0,031	861472	9685	828528	315	6234	10	5198	6	0,969
0,032	851787	9675	828213	325	6224	11	5192	5	0,968
0,033	842112	9665	827888	335	6213	10	5187	5	0,967
0,034	832447	9655	827553	345	6203	11	5182	6	0,966
0,035	822792	9645	827208	355	6192	11	5176	6	0,965
0,036	813147	9635	826853	365	6181	12	5170	6	0,964
0,037	803512	9625	826488	375	6169	12	5164	6	0,963
0,038	793887	9615	826113	385	6157	12	5158	7	0,962
0,039	784272	9605	825728	395	6145	12	5151	6	0,961
0,040	−0,3774667+	9595	−0,0825333+	405	+0,026133−	13	+0,015145−	7	0,960
0,041	765072	9585	824928	415	6120	13	5138	7	0,959
0,042	755487	9575	824513	425	6107	13	5131	7	0,958
0,043	745912	9565	824088	435	6094	14	5124	7	0,957
0,044	736347	9555	823653	445	6080	14	5117	8	0,956
0,045	726792	9545	823208	455	6066	14	5109	7	0,955
0,046	717247	9535	822753	465	6052	14	5102	8	0,954
0,047	707712	9525	822288	475	6038	15	5094	8	0,953
0,048	698187	9515	821813	485	6023	15	5086	8	0,952
0,049	688672	9505	821328	495	6008	15	5078	8	0,951
0,050	−0,3679167+		−0,0820833+		+0,025993−		+0,015070−		0,950

m	$''E_1$		$''E_0$		$''E_1''$		$''E''$		n
0,050	+0,0637292+	3675	−0,0041458−	821	−0,002854−	26	+0,000760+	15	0,950
0,051	633617	3665	042279	820	2828	26	0775	16	0,949
0,052	629952	3655	043099	820	2802	26	0791	15	0,948
0,053	626297	3646	043919	819	2776	26	0806	15	0,947
0,054	622651	3637	044738	818	2750	26	0821	15	0,946
0,055	619014	3627	045556	818	2724	26	0836	15	0,945
0,056	615387	3617	046374	817	2698	26	0851	15	0,944
0,057	611770	3609	047191	817	2672	26	0866	15	0,943
0,058	608161	3598	048008	816	2646	25	0881	15	0,942
0,059	604563	3590	048824	816	2621	26	0896	15	0,941
0,060	+0,0600973+	3580	−0,0049640−	815	−0,002595−	26	+0,000911+	15	0,940
0,061	597393	3570	050455	814	2569	26	0926	15	0,939
0,062	593823	3561	051269	814	2543	26	0941	15	0,938
0,063	590262	3552	052083	813	2517	25	0956	15	0,937
0,064	586710	3543	052896	813	2492	26	0971	14	0,936
0,065	583167	3533	053709	812	2466	26	0985	15	0,935
0,066	579634	3524	054521	811	2440	26	1000	15	0,934
0,067	576110	3514	055332	811	2414	25	1015	15	0,933
0,068	572596	3505	056143	809	2389	26	1030	15	0,932
0,069	569091	3496	056952	810	2363	25	1045	15	0,931
0,070	+0,0565595+	3487	−0,0057762−	808	−0,002338−	26	+0,001060+	15	0,930
0,071	562108	3477	058570	808	2312	26	1075	15	0,929
0,072	558631	3468	059378	807	2286	25	1090	14	0,928
0,073	555163	3458	060185	806	2261	26	1104	15	0,927
0,074	551705	3450	060991	806	2235	25	1119	15	0,926
0,075	548255	3440	061797	805	2210	26	1134	15	0,925
0,076	544815	3431	062602	804	2184	25	1149	15	0,924
0,077	541384	3422	063406	803	2159	26	1164	15	0,923
0,078	537962	3412	064209	803	2133	25	1179	14	0,922
0,079	534550	3403	065012	801	2108	26	1193	15	0,921
0,080	+0,0531147+	3394	−0,0065813−	801	−0,002082−	25	+0,001208+	15	0,920
0,081	527753	3385	066614	800	2057	25	1223	14	0,919
0,082	524368	3376	067414	800	2032	26	1237	15	0,918
0,083	520992	3367	068214	798	2006	25	1252	15	0,917
0,084	517625	3357	069012	798	1981	25	1267	15	0,916
0,085	514268	3348	069810	797	1956	26	1282	14	0,915
0,086	510920	3339	070607	795	1930	25	1296	15	0,914
0,087	507581	3330	071402	796	1905	25	1311	15	0,913
0,088	504251	3321	072198	794	1880	25	1326	14	0,912
0,089	500930	3312	072992	793	1855	26	1340	15	0,911
0,090	+0,0497618+	3302	−0,0073785−	792	−0,001829−	25	+0,001355+	14	0,910
0,091	494316	3294	074577	792	1804	25	1369	15	0,909
0,092	491022	3284	075369	790	1779	25	1384	15	0,908
0,093	487738	3276	076159	790	1754	25	1399	14	0,907
0,094	484462	3266	076949	789	1729	25	1413	15	0,906
0,095	481196	3257	077738	787	1704	25	1428	14	0,905
0,096	477939	3248	078525	787	1679	25	1442	15	0,904
0,097	474691	3240	079312	786	1654	25	1457	14	0,903
0,098	471451	3230	080098	785	1629	25	1471	15	0,902
0,099	468221	3221	080883	784	1604	25	1486	14	0,901
0,100	+0,0465000+		−0,0081667−		−0,001579−		+0,001500+		0,900

m	$'E_1$		$'E_0$		$'E_1''$		$'E_0''$		n
0,050	−0,3679167+	9495	−0,0820833+	505	+0,025993−	16	+0,015070−	9	0,950
0,051	669672	9485	820328	515	5977	16	5061	8	0,949
0,052	660187	9475	819813	525	5961	16	5053	9	0,948
0,053	650712	9465	819288	535	5945	16	5044	9	0,947
0,054	641247	9455	818753	545	5929	17	5035	9	0,946
0,055	631792	9445	818208	555	5912	17	5026	9	0,945
0,056	622347	9435	817653	565	5895	17	5017	10	0,944
0,057	612912	9425	817088	575	5878	18	5007	9	0,943
0,058	603487	9415	816513	585	5860	18	4998	10	0,942
0,059	594072	9405	815928	595	5842	18	4988	10	0,941
0,060	−0,3584667+	9395	−0,0815333+	605	+0,025824−	18	+0,014978−	10	0,940
0,061	575272	9385	814728	615	5806	19	4968	10	0,939
0,062	565887	9375	814113	625	5787	19	4958	10	0,938
0,063	556512	9365	813488	635	5768	19	4948	11	0,937
0,064	547147	9355	812853	645	5749	19	4937	11	0,936
0,065	537792	9345	812208	655	5730	20	4926	10	0,935
0,066	528447	9335	811553	665	5710	20	4916	11	0,934
0,067	519112	9325	810888	675	5690	20	4905	12	0,933
0,068	509787	9315	810213	685	5670	21	4893	11	0,932
0,069	500472	9305	809528	695	5649	21	4882	12	0,931
0,070	−0,3491167+	9295	−0,0808833+	705	+0,025628−	21	+0,014870−	11	0,930
0,071	481872	9285	808128	715	5607	21	4859	12	0,929
0,072	472587	9275	807413	725	5586	22	4847	12	0,928
0,073	463312	9265	806688	735	5564	21	4835	12	0,927
0,074	454047	9255	805953	745	5543	23	4823	13	0,926
0,075	444792	9245	805208	755	5520	22	4810	12	0,925
0,076	435547	9235	804453	765	5498	23	4798	13	0,924
0,077	426312	9225	803688	775	5475	23	4785	13	0,923
0,078	417087	9215	802913	785	5452	23	4772	13	0,922
0,079	407872	9205	802128	795	5429	23	4759	13	0,921
0,080	−0,3398667+	9195	−0,0801333+	805	+0,025406−	24	+0,014746−	13	0,920
0,081	389472	9185	800528	815	5382	24	4733	14	0,919
0,082	380287	9175	799713	825	5358	24	4719	13	0,918
0,083	371112	9165	798888	835	5334	24	4706	14	0,917
0,084	361947	9155	798053	845	5310	25	4692	14	0,916
0,085	352792	9145	797208	855	5285	25	4678	14	0,915
0,086	343647	9135	796353	865	5260	25	4664	15	0,914
0,087	334512	9125	795488	875	5235	26	4649	14	0,913
0,088	325387	9115	794613	885	5209	25	4635	15	0,912
0,089	316272	9105	793728	895	5184	26	4620	14	0,911
0,090	−0,3307167+	9095	−0,0792833+	905	+0,025158−	27	+0,014606−	15	0,910
0,091	298072	9085	791928	915	5131	26	4591	16	0,909
0,092	288987	9075	791013	925	5105	27	4575	15	0,908
0,093	279912	9065	790088	935	5078	27	4560	15	0,907
0,094	270847	9055	789153	945	5051	27	4545	16	0,906
0,095	261792	9045	788208	955	5024	27	4529	16	0,905
0,096	252747	9035	787253	965	4997	28	4513	16	0,904
0,097	243712	9025	786288	975	4969	28	4497	16	0,903
0,098	234687	9015	785313	985	4941	28	4481	16	0,902
0,099	225672	9005	784328	995	4913	28	4465	16	0,901
0,100	−0,3216667+		−0,0783333+		+0,024885−		+0,014449−		0,900

Table VI (cont'd)

n	$''E_0$	Δ	$''E_1$	Δ	$''E_0''$	Δ	$''E_1''$	Δ	m
0.100	+0.0465000+	3212	−0.0081667−	782	−0.001579−	25	+0.001500+	15	0.900
0.101	461788	3203	082449	782	1554	24	1515	14	0.899
0.102	458585	3195	083231	781	1530	25	1529	14	0.898
0.103	455390	3185	084012	780	1505	25	1543	15	0.897
0.104	452205	3176	084792	779	1480	25	1558	14	0.896
0.105	449029	3167	085571	777	1455	25	1572	14	0.895
0.106	445862	3159	086348	777	1430	24	1586	15	0.894
0.107	442703	3149	087125	775	1406	25	1601	14	0.893
0.108	439554	3141	087900	775	1381	25	1615	14	0.892
0.109	436413	3131	088675	773	1356	24	1629	15	0.891
0.110	+0.0433282+	3123	−0.0089448−	773	−0.001332−	25	+0.001644+	14	0.890
0.111	430159	3114	090221	771	1307	24	1658	14	0.889
0.112	427045	3105	090992	770	1283	25	1672	14	0.888
0.113	423940	3096	091762	769	1258	24	1686	15	0.887
0.114	420844	3087	092531	768	1234	25	1701	14	0.886
0.115	417757	3078	093299	766	1209	24	1715	14	0.885
0.116	414679	3070	094065	766	1185	24	1729	14	0.884
0.117	411609	3061	094831	764	1161	25	1743	14	0.883
0.118	408548	3052	095595	763	1136	24	1757	14	0.882
0.119	405496	3043	096358	762	1112	24	1771	15	0.881
0.120	+0.0402453+	3034	−0.0097120−	761	−0.001088−	25	+0.001786+	14	0.880
0.121	399419	3025	097881	759	1063	24	1800	14	0.879
0.122	396394	3017	098640	759	1039	24	1814	14	0.878
0.123	393377	3008	099399	757	1015	24	1828	14	0.877
0.124	390369	2999	100156	755	0991	24	1842	14	0.876
0.125	387370	2991	100911	755	0967	24	1856	14	0.875
0.126	384379	2981	101666	753	0943	24	1870	14	0.874
0.127	381398	2973	102419	752	0919	25	1884	14	0.873
0.128	378425	2964	103171	751	0894	23	1898	14	0.872
0.129	375461	2956	103922	750	0871	24	1912	13	0.871
0.130	+0.0372505+	2947	−0.0104672−	748	−0.000847−	24	+0.001925+	14	0.870
0.131	369558	2938	105420	747	0823	24	1939	14	0.869
0.132	366620	2929	106167	745	0799	24	1953	14	0.868
0.133	363691	2921	106912	744	0775	24	1967	14	0.867
0.134	360770	2912	107656	743	0751	24	1981	14	0.866
0.135	357858	2904	108399	742	0727	23	1995	13	0.865
0.136	354954	2895	109141	740	0704	24	2008	14	0.864
0.137	352059	2886	109881	739	0680	24	2022	14	0.863
0.138	349173	2877	110620	737	0656	23	2036	13	0.862
0.139	346296	2869	111357	736	0633	24	2049	14	0.861
0.140	+0.0343427+	2861	−0.0112093−	735	−0.000609−	23	+0.002063+	14	0.860
0.141	340566	2851	112828	733	0586	24	2077	13	0.859
0.142	337715	2844	113561	732	0562	23	2090	14	0.858
0.143	334871	2834	114293	730	0539	24	2104	14	0.857
0.144	332037	2826	115023	729	0515	23	2118	13	0.856
0.145	329211	2818	115752	728	0492	24	2131	14	0.855
0.146	326393	2809	116480	726	0468	23	2145	13	0.854
0.147	323584	2800	117206	724	0445	23	2158	14	0.853
0.148	320784	2792	117930	723	0422	23	2172	13	0.852
0.149	317992	2784	118653	722	0399	24	2185	14	0.851
0.150	+0.0315208+		−0.0119375−		−0.000375−		+0.002199+		0.850

n	$'E_0$	Δ	$'E_1$	Δ	$'E_0''$	Δ	$'E_1''$	Δ	m
0.100	−0.3216667+	8995	−0.0783333+	1005	+0.024885−	29	+0.014449−	17	0.900
0.101	207672	8985	782328	1015	4856	29	4432	17	0.899
0.102	198687	8975	781313	1025	4827	29	4415	17	0.898
0.103	189712	8965	780288	1035	4798	29	4398	17	0.897
0.104	180747	8955	779253	1045	4769	30	4381	17	0.896
0.105	171792	8945	778208	1055	4739	30	4364	17	0.895
0.106	162847	8935	777153	1065	4709	30	4347	18	0.894
0.107	153912	8925	776088	1075	4679	30	4329	18	0.893
0.108	144987	8915	775013	1085	4649	30	4311	17	0.892
0.109	136072	8905	773928	1095	4619	31	4294	18	0.891
0.110	−0.3127167+	8895	−0.0772833+	1105	+0.024588−	31	+0.014276−	19	0.890
0.111	118272	8885	771728	1115	4557	31	4257	18	0.889
0.112	109387	8875	770613	1125	4526	32	4239	19	0.888
0.113	100512	8865	769488	1135	4494	31	4220	18	0.887
0.114	091647	8855	768353	1145	4463	32	4202	19	0.886
0.115	082792	8845	767208	1155	4431	32	4183	19	0.885
0.116	073947	8835	766053	1165	4399	32	4164	19	0.884
0.117	065112	8825	764888	1175	4367	33	4145	19	0.883
0.118	056287	8815	763713	1185	4334	33	4126	20	0.882
0.119	047472	8805	762528	1195	4301	33	4106	20	0.881
0.120	−0.3038667+	8795	−0.0761333+	1205	+0.024268−	33	+0.014086−	19	0.880
0.121	029872	8785	760128	1215	4235	33	4067	20	0.879
0.122	021087	8775	758913	1225	4202	34	4047	20	0.878
0.123	012312	8765	757688	1235	4168	34	4027	21	0.877
0.124	−0.3003547	8755	756453	1245	4134	34	4006	20	0.876
0.125	−0.2994792	8745	755208	1255	4100	34	3986	21	0.875
0.126	986047	8735	753953	1265	4066	35	3965	20	0.874
0.127	977312	8725	752688	1275	4031	34	3945	21	0.873
0.128	968587	8715	751413	1285	3997	35	3924	21	0.872
0.129	959872	8705	750128	1295	3962	36	3903	22	0.871
0.130	−0.2951167+	8695	−0.0748833+	1305	+0.023926−	35	+0.013881−	21	0.870
0.131	942472	8685	747528	1315	3891	35	3860	22	0.869
0.132	933787	8675	746213	1325	3856	36	3838	21	0.868
0.133	925112	8665	744888	1335	3820	36	3817	22	0.867
0.134	916447	8655	743553	1345	3784	36	3795	22	0.866
0.135	907792	8645	742208	1355	3748	37	3773	22	0.865
0.136	899147	8635	740853	1365	3711	36	3751	23	0.864
0.137	890512	8625	739488	1375	3675	37	3728	22	0.863
0.138	881887	8615	738113	1385	3638	37	3706	23	0.862
0.139	873272	8605	736728	1395	3601	37	3683	23	0.861
0.140	−0.2864667+	8595	−0.0735333+	1405	+0.023564−	38	+0.013660−	23	0.860
0.141	856072	8585	733928	1415	3526	38	3637	23	0.859
0.142	847487	8575	732513	1425	3488	37	3614	23	0.858
0.143	838912	8565	731088	1435	3451	38	3591	23	0.857
0.144	830347	8555	729653	1445	3413	39	3568	24	0.856
0.145	821792	8545	728208	1455	3374	38	3544	24	0.855
0.146	813247	8535	726753	1465	3336	39	3520	24	0.854
0.147	804712	8525	725288	1475	3297	38	3496	24	0.853
0.148	796187	8515	723813	1485	3259	39	3472	24	0.852
0.149	787672	8505	722328	1495	3220	40	3448	24	0.851
0.150	−0.2779167+		−0.0720833+		+0.023180−		+0.013424−		0.850

Table VI (cont'd)

m	$''E_1$		$''E_0$		$''E_1''$		$''E_0''$		n
0,150	+0,0315208+	2775	−0,0119375−	720	−0,000375−	23	+0,002199+	13	0,850
0,151	312433	2766	120095	719	0352	23	2212	13	0,849
0,152	309667	2758	120814	717	0329	23	2225	14	0,848
0,153	306909	2749	121531	715	0306	23	2239	13	0,847
0,154	304160	2741	122246	714	0283	23	2252	13	0,846
0,155	301419	2733	122960	713	0260	23	2265	14	0,845
0,156	298686	2724	123673	711	0237	23	2279	13	0,844
0,157	295962	2716	124384	709	0214	23	2292	13	0,843
0,158	293246	2707	125093	708	0191	23	2305	13	0,842
0,159	290539	2699	125801	706	0168	22	2318	14	0,841
0,160	+0,0287840+	2690	−0,0126507−	704	−0,000146−	23	+0,002332+	13	0,840
0,161	285150	2683	127211	703	0123	23	2345	13	0,839
0,162	282467	2673	127914	701	0100	23	2358	13	0,838
0,163	279794	2666	128615	700	0077	22	2371	13	0,837
0,164	277128	2657	129315	698	0055	23	2384	13	0,836
0,165	274471	2648	130013	697	0032	22	2397	13	0,835
0,166	271823	2640	130710	694	−0,000010−	23	2410	13	0,834
0,167	269183	2632	131404	693	+0,000013+	22	2423	13	0,833
0,168	266551	2624	132097	692	0035	23	2436	13	0,832
0,169	263927	2615	132789	689	0058	22	2449	13	0,831
0,170	+0,0261312+	2607	−0,0133478−	688	+0,000080+	22	+0,002462+	13	0,830
0,171	258705	2599	134166	687	0102	23	2475	13	0,829
0,172	256106	2591	134853	684	0125	22	2488	13	0,828
0,173	253515	2582	135537	683	0147	22	2501	12	0,827
0,174	250933	2574	136220	681	0169	22	2513	13	0,826
0,175	248359	2565	136901	679	0191	22	2526	13	0,825
0,176	245794	2558	137580	678	0213	23	2539	13	0,824
0,177	243236	2549	138258	676	0236	22	2552	12	0,823
0,178	240687	2541	138934	674	0258	22	2564	13	0,822
0,179	238146	2533	139608	672	0280	21	2577	13	0,821
0,180	+0,0235613+	2524	−0,0140280−	670	+0,000301+	22	+0,002590+	12	0,820
0,181	233089	2517	140950	669	0323	22	2602	13	0,819
0,182	230572	2508	141619	667	0345	22	2615	12	0,818
0,183	228064	2500	142286	665	0367	22	2627	13	0,817
0,184	225564	2492	142951	663	0389	22	2640	12	0,816
0,185	223072	2483	143614	661	0411	21	2652	13	0,815
0,186	220589	2476	144275	660	0432	22	2665	12	0,814
0,187	218113	2467	144935	657	0454	21	2677	13	0,813
0,188	215646	2460	145592	656	0475	22	2690	12	0,812
0,189	213186	2451	146248	654	0497	21	2702	12	0,811
0,190	+0,0210735+	2443	−0,0146902−	652	+0,000518+	22	+0,002714+	13	0,810
0,191	208292	2435	147554	650	0540	21	2727	12	0,809
0,192	205857	2427	148204	648	0561	22	2739	12	0,808
0,193	203430	2419	148852	646	0583	21	2751	12	0,807
0,194	201011	2411	149498	644	0604	21	2763	13	0,806
0,195	198600	2403	150142	642	0625	21	2776	12	0,805
0,196	196197	2394	150784	640	0646	21	2788	12	0,804
0,197	193803	2387	151424	639	0667	22	2800	12	0,803
0,198	191416	2379	152063	636	0689	21	2812	12	0,802
0,199	189037	2370	152699	634	0710	21	2824	12	0,801
0,200	+0,0186667+		−0,0153333−		+0,000731+		+0,002836+		0,800

m	$'E_1$		$'E_0$		$'E_1''$		$'E_0''$		n
0,150	−0,2779167+	8495	−0,0720833+	1505	+0,023180−	39	+0,013424−	25	0,850
0,151	770672	8485	719328	1515	3141	40	3399	24	0,849
0,152	762187	8475	717813	1525	3101	40	3375	25	0,848
0,153	753712	8465	716288	1535	3061	40	3350	25	0,847
0,154	745247	8455	714753	1545	3021	40	3325	25	0,846
0,155	736792	8445	713208	1555	2981	40	3300	26	0,845
0,156	728347	8435	711653	1565	2941	41	3274	25	0,844
0,157	719912	8425	710088	1575	2900	40	3249	26	0,843
0,158	711487	8415	708513	1585	2860	41	3223	25	0,842
0,159	703072	8405	706928	1595	2819	41	3198	26	0,841
0,160	−0,2694667+	8395	−0,0705333+	1605	+0,022778−	42	+0,013172−	26	0,840
0,161	686272	8385	703728	1615	2736	41	3146	27	0,839
0,162	677887	8375	702113	1625	2695	42	3119	26	0,838
0,163	669512	8365	700488	1635	2653	42	3093	26	0,837
0,164	661147	8355	698853	1645	2611	42	3067	27	0,836
0,165	652792	8345	697208	1655	2569	42	3040	27	0,835
0,166	644447	8335	695553	1665	2527	42	3013	27	0,834
0,167	636112	8325	693888	1675	2485	43	2986	27	0,833
0,168	627787	8315	692213	1685	2442	43	2959	27	0,832
0,169	619472	8305	690528	1695	2399	43	2932	28	0,831
0,170	−0,2611167+	8295	−0,0688833+	1705	+0,022356−	43	+0,012904−	27	0,830
0,171	602872	8285	687128	1715	2313	43	2877	28	0,829
0,172	594587	8275	685413	1725	2270	44	2849	28	0,828
0,173	586312	8265	683688	1735	2226	43	2821	28	0,827
0,174	578047	8255	681953	1745	2183	44	2793	28	0,826
0,175	569792	8245	680208	1755	2139	44	2765	29	0,825
0,176	561547	8235	678453	1765	2095	44	2736	28	0,824
0,177	553312	8225	676688	1775	2051	45	2708	29	0,823
0,178	545087	8215	674913	1785	2006	44	2679	29	0,822
0,179	536872	8205	673128	1795	1962	45	2650	28	0,821
0,180	−0,2528667+	8195	−0,0671333+	1805	+0,021917−	45	+0,012622−	30	0,820
0,181	520472	8185	669528	1815	1872	45	2592	29	0,819
0,182	512287	8175	667713	1825	1827	45	2563	29	0,818
0,183	504112	8165	665888	1835	1782	45	2534	30	0,817
0,184	495947	8155	664053	1845	1737	46	2504	29	0,816
0,185	487792	8145	662208	1855	1691	46	2475	30	0,815
0,186	479647	8135	660353	1865	1645	45	2445	30	0,814
0,187	471512	8125	658488	1875	1600	46	2415	31	0,813
0,188	463387	8115	656613	1885	1554	47	2384	30	0,812
0,189	455272	8105	654728	1895	1507	46	2354	30	0,811
0,190	−0,2447167+	8095	−0,0652833+	1905	+0,021461−	46	+0,012324−	31	0,810
0,191	439072	8085	650928	1915	1415	47	2293	31	0,809
0,192	430987	8075	649013	1925	1368	47	2262	30	0,808
0,193	422912	8065	647088	1935	1321	47	2232	32	0,807
0,194	414847	8055	645153	1945	1274	47	2200	31	0,806
0,195	406792	8045	643208	1955	1227	47	2169	31	0,805
0,196	398747	8035	641253	1965	1180	48	2138	32	0,804
0,197	390712	8025	639288	1975	1132	47	2106	31	0,803
0,198	382687	8015	637313	1985	1085	48	2075	32	0,802
0,199	374672	8005	635328	1995	1037	48	2043	32	0,801
0,200	−0,2366667+		−0,0633333+		+0,020989−		+0,012011−		0,800

Table VI (cont'd)

n	"E_0	Δ	"E_1	Δ	"E_0''	Δ	"E_1''	Δ	m	n	'E_0	Δ	'E_1	Δ	'E_0''	Δ	'E_1''	Δ	m
0.200	+0.0186667+	2363	−0.0153333−	633	+0.000731+	21	+0.002836+	12	0.800	0.200	−0.2366667+	7995	−0.0633333+	2005	+0.020989−	48	+0.012011−	32	0.800
0.201	184304	2355	153966	630	0752	21	2848	12	0.799	0.201	358672	7985	631328	2015	0941	48	1979	32	0.799
0.202	181949	2346	154596	628	0773	20	2860	12	0.798	0.202	350687	7975	629313	2025	0893	49	1947	33	0.798
0.203	179603	2339	155224	627	0793	21	2872	12	0.797	0.203	342712	7965	627288	2035	0844	48	1914	32	0.797
0.204	177264	2331	155851	624	0814	21	2884	12	0.796	0.204	334747	7955	625253	2045	0796	49	1882	33	0.796
0.205	174933	2323	156475	622	0835	21	2896	11	0.795	0.205	326792	7945	623208	2055	0747	49	1849	33	0.795
0.206	172610	2315	157097	620	0856	20	2907	12	0.794	0.206	318847	7935	621153	2065	0698	49	1816	32	0.794
0.207	170295	2307	157717	618	0876	21	2919	12	0.793	0.207	310912	7925	619088	2075	0649	49	1784	34	0.793
0.208	167988	2299	158335	616	0897	21	2931	12	0.792	0.208	302987	7915	617013	2085	0600	49	1750	33	0.792
0.209	165689	2291	158951	614	0918	20	2943	11	0.791	0.209	295072	7905	614928	2095	0551	50	1717	33	0.791
0.210	+0.0163398+	2283	−0.0159565−	612	+0.000938+	21	+0.002954+	12	0.790	0.210	−0.2287167+	7895	−0.0612833+	2105	+0.020501−	49	+0.011684−	34	0.790
0.211	161115	2275	160177	609	0959	20	2966	12	0.789	0.211	279272	7885	610728	2115	0452	50	1650	33	0.789
0.212	158840	2268	160786	608	0979	20	2978	11	0.788	0.212	271387	7875	608613	2125	0402	50	1617	34	0.788
0.213	156572	2259	161394	605	0999	21	2989	12	0.787	0.213	263512	7865	606488	2135	0352	50	1583	34	0.787
0.214	154313	2252	161999	604	1020	20	3001	11	0.786	0.214	255647	7855	604353	2145	0302	50	1549	34	0.786
0.215	152061	2244	162603	601	1040	20	3012	12	0.785	0.215	247792	7845	602208	2155	0252	50	1515	35	0.785
0.216	149817	2236	163204	599	1060	20	3024	11	0.784	0.216	239947	7835	600053	2165	0202	51	1480	34	0.784
0.217	147581	2228	163803	597	1080	21	3035	12	0.783	0.217	232112	7825	597888	2175	0151	50	1446	34	0.783
0.218	145353	2220	164400	594	1101	20	3047	11	0.782	0.218	224287	7815	595713	2185	0101	51	1412	35	0.782
0.219	143133	2213	164994	593	1121	20	3058	12	0.781	0.219	216472	7805	593528	2195	+0.020050	51	1377	35	0.781
0.220	+0.0140920+	2205	−0.0165587−	590	+0.001141+	20	+0.003070+	11	0.780	0.220	−0.2208667+	7795	−0.0591333+	2205	+0.019999−	51	+0.011342−	35	0.780
0.221	138715	2197	166177	588	1161	20	3081	11	0.779	0.221	200872	7785	589128	2215	9948	51	1307	35	0.779
0.222	136518	2189	166765	586	1181	19	3092	11	0.778	0.222	193087	7775	586913	2225	9897	51	1272	35	0.778
0.223	134329	2181	167351	583	1200	20	3103	12	0.777	0.223	185312	7765	584688	2235	9846	51	1237	36	0.777
0.224	132148	2174	167934	582	1220	20	3115	11	0.776	0.224	177547	7755	582453	2245	9795	52	1201	35	0.776
0.225	129974	2166	168516	579	1240	20	3126	11	0.775	0.225	169792	7745	580208	2255	9743	52	1166	36	0.775
0.226	127808	2158	169095	577	1260	19	3137	11	0.774	0.226	162047	7735	577953	2265	9691	51	1130	36	0.774
0.227	125650	2151	169672	574	1279	20	3148	11	0.773	0.227	154312	7725	575688	2275	9640	52	1094	36	0.773
0.228	123499	2142	170246	572	1299	20	3159	11	0.772	0.228	146587	7715	573413	2285	9588	52	1058	36	0.772
0.229	121357	2135	170818	570	1319	19	3170	11	0.771	0.229	138872	7705	571128	2295	9536	53	1022	36	0.771
0.230	+0.0119222+	2128	−0.0171388−	568	+0.001338+	20	+0.003181+	11	0.770	0.230	−0.2131167+	7695	−0.0568833+	2305	+0.019483−	52	+0.010986−	36	0.770
0.231	117094	2119	171956	565	1358	19	3192	11	0.769	0.231	123472	7685	566528	2315	9431	52	0950	37	0.769
0.232	114975	2112	172521	563	1377	19	3203	11	0.768	0.232	115787	7675	564213	2325	9379	53	0913	37	0.768
0.233	112863	2105	173084	561	1396	20	3214	11	0.767	0.233	108112	7665	561888	2335	9326	53	0876	36	0.767
0.234	110758	2096	173645	559	1416	19	3225	11	0.766	0.234	100447	7655	559553	2345	9273	52	0840	37	0.766
0.235	108662	2089	174204	556	1435	19	3236	11	0.765	0.235	092792	7645	557208	2355	9221	53	0803	37	0.765
0.236	106573	2081	174760	553	1454	19	3247	10	0.764	0.236	085147	7635	554853	2365	9168	53	0766	38	0.764
0.237	104492	2074	175313	552	1473	19	3257	11	0.763	0.237	077512	7625	552488	2375	9115	54	0728	37	0.763
0.238	102418	2066	175865	548	1492	19	3268	11	0.762	0.238	069887	7615	550113	2385	9061	53	0691	37	0.762
0.239	100352	2059	176413	547	1511	19	3279	10	0.761	0.239	062272	7605	547728	2395	9008	53	0654	38	0.761
0.240	+0.0098293+	2051	−0.0176960−	544	+0.001530+	19	+0.003289+	11	0.760	0.240	−0.2054667+	7595	−0.0545333+	2405	+0.018955−	54	+0.010616−	38	0.760
0.241	096242	2043	177504	542	1549	19	3300	10	0.759	0.241	047072	7585	542928	2415	8901	54	0578	38	0.759
0.242	094199	2036	178046	539	1568	19	3310	11	0.758	0.242	039487	7575	540513	2425	8847	53	0540	38	0.758
0.243	092163	2028	178585	537	1587	19	3321	10	0.757	0.243	031912	7565	538088	2435	8794	54	0502	38	0.757
0.244	090135	2020	179122	534	1606	18	3331	11	0.756	0.244	024347	7555	535653	2445	8740	54	0464	38	0.756
0.245	088115	2013	179656	532	1624	19	3342	10	0.755	0.245	016792	7545	533208	2455	8686	55	0426	39	0.755
0.246	086102	2006	180188	530	1643	19	3352	11	0.754	0.246	009247	7535	530753	2465	8631	54	0387	38	0.754
0.247	084096	1998	180718	527	1662	18	3363	10	0.753	0.247	−0.2001712	7525	528288	2475	8577	54	0349	39	0.753
0.248	082098	1990	181245	525	1680	19	3373	10	0.752	0.248	−0.1994187	7515	525813	2485	8523	55	0310	39	0.752
0.249	080108	1983	181770	522	1699	18	3383	11	0.751	0.249	986672	7505	523328	2495	8468	54	0271	39	0.751
0.250	+0.0078125+		−0.0182292−		+0.001717+		+0.003394+		0.750	0.250	−0.1979167+		−0.0520833+		+0.018414−		+0.010232−		0.750

m	"E₁		"E₀		"E₁''		"E₀''		n	m	'E₁		'E₀		'E₁''		'E₀''		n
0,250	+0,0078125+	1975	−0,0182292−	519	+0,001717+	19	+0,003394+	10	0,750	0,250	−0,1979167+	7495	−0,0520833+	2505	+0,018414−	55	+0,010232−	39	0,750
0,251	076150	1968	182811	517	1736	18	3404	10	0,749	0,251	971672	7485	518328	2515	8359	55	0193	39	0,749
0,252	074182	1961	183328	515	1754	18	3414	10	0,748	0,252	964187	7475	515813	2525	8304	55	0154	40	0,748
0,253	072221	1953	183843	512	1772	18	3424	10	0,747	0,253	956712	7465	513288	2535	8249	55	0114	39	0,747
0,254	070268	1945	184355	509	1790	19	3434	10	0,746	0,254	949247	7455	510753	2545	8194	55	0075	40	0,746
0,255	068323	1938	184864	507	1809	18	3444	10	0,745	0,255	941792	7445	508208	2555	8139	56	+0,010035	40	0,745
0,256	066385	1931	185371	505	1827	18	3454	10	0,744	0,256	934347	7435	505653	2565	8083	55	+0,009995	40	0,744
0,257	064454	1923	185876	501	1845	18	3464	10	0,743	0,257	926912	7425	503088	2575	8028	55	9955	40	0,743
0,258	062531	1916	186377	500	1863	18	3474	10	0,742	0,258	919487	7415	500513	2585	7973	56	9915	40	0,742
0,259	060615	1908	186877	496	1881	18	3484	10	0,741	0,259	912072	7405	497928	2595	7917	56	9875	40	0,741
0,260	+0,0058707+	1901	−0,0187373−	494	+0,001899+	17	+0,003494+	10	0,740	0,260	−0,1904667+	7395	−0,0495333+	2605	+0,017861−	56	+0,009835−	41	0,740
0,261	056806	1894	187867	492	1916	18	3504	9	0,739	0,261	897272	7385	492728	2615	7805	56	9794	40	0,739
0,262	054912	1886	188359	489	1934	18	3513	10	0,738	0,262	889887	7375	490113	2625	7749	56	9754	41	0,738
0,263	053026	1879	188848	486	1952	18	3523	10	0,737	0,263	882512	7365	487488	2635	7693	56	9713	41	0,737
0,264	051147	1871	189334	483	1970	17	3533	10	0,736	0,264	875147	7355	484853	2645	7637	56	9672	41	0,736
0,265	049276	1864	189817	481	1987	18	3543	9	0,735	0,265	867792	7345	482208	2655	7581	57	9631	41	0,735
0,266	047412	1857	190298	478	2005	17	3552	10	0,734	0,266	860447	7335	479553	2665	7524	56	9590	41	0,734
0,267	045555	1850	190776	476	2022	18	3562	9	0,733	0,267	853112	7325	476888	2675	7468	57	9549	42	0,733
0,268	043705	1842	191252	473	2040	17	3571	10	0,732	0,268	845787	7315	474213	2685	7411	56	9507	41	0,732
0,269	041863	1835	191725	470	2057	17	3581	9	0,731	0,269	838472	7305	471528	2695	7355	57	9466	42	0,731
0,270	+0,0040028+	1827	−0,0192195−	467	+0,002074+	18	+0,003590+	10	0,730	0,270	−0,1831167+	7295	−0,0468833+	2705	+0,017298−	57	+0,009424−	42	0,730
0,271	038201	1820	192662	465	2092	17	3600	9	0,729	0,271	823872	7285	466128	2715	7241	57	9382	41	0,729
0,272	036381	1813	193127	462	2109	17	3609	9	0,728	0,272	816587	7275	463413	2725	7184	57	9341	43	0,728
0,273	034568	1806	193589	460	2126	17	3618	10	0,727	0,273	809312	7265	460688	2735	7127	57	9298	42	0,727
0,274	032762	1798	194049	456	2143	17	3628	9	0,726	0,274	802047	7255	457953	2745	7070	57	9256	42	0,726
0,275	030964	1792	194505	454	2160	17	3637	9	0,725	0,275	794792	7245	455208	2755	7013	58	9214	42	0,725
0,276	029172	1784	194959	451	2177	17	3646	9	0,724	0,276	787547	7235	452453	2765	6955	57	9172	43	0,724
0,277	027388	1776	195410	448	2194	17	3655	9	0,723	0,277	780312	7225	449688	2775	6898	58	9129	43	0,723
0,278	025612	1770	195858	446	2211	17	3664	9	0,722	0,278	773087	7215	446913	2785	6840	57	9086	43	0,722
0,279	023842	1762	196304	443	2228	16	3673	9	0,721	0,279	765872	7205	444128	2795	6783	58	9043	42	0,721
0,280	+0,0022080+	1755	−0,0196747−	440	+0,002244+	17	+0,003682+	9	0,720	0,280	−0,1758667+	7195	−0,0441333+	2805	+0,016725−	58	+0,009001−	44	0,720
0,281	020325	1748	197187	437	2261	17	3691	9	0,719	0,281	751472	7185	438528	2815	6667	58	8957	43	0,719
0,282	018577	1741	197624	434	2278	16	3700	9	0,718	0,282	744287	7175	435713	2825	6609	58	8914	43	0,718
0,283	016836	1733	198058	431	2294	17	3709	9	0,717	0,283	737112	7165	432888	2835	6551	58	8871	43	0,717
0,284	015103	1727	198489	429	2311	16	3718	9	0,716	0,284	729947	7155	430053	2845	6493	58	8828	44	0,716
0,285	013376	1719	198918	426	2327	17	3727	9	0,715	0,285	722792	7145	427208	2855	6435	59	8784	44	0,715
0,286	011657	1712	199344	423	2344	16	3736	8	0,714	0,286	715647	7135	424353	2865	6376	58	8740	44	0,714
0,287	009945	1705	199767	420	2360	16	3744	9	0,713	0,287	708512	7125	421488	2875	6318	58	8696	44	0,713
0,288	008240	1698	200187	417	2376	17	3753	9	0,712	0,288	701387	7115	418613	2885	6260	59	8652	44	0,712
0,289	006542	1690	200604	414	2393	16	3762	8	0,711	0,289	694272	7105	415728	2895	6201	59	8608	44	0,711
0,290	+0,0004852+	1684	−0,0201018−	412	+0,002409+	16	+0,003770+	9	0,710	0,290	−0,1687167+	7095	−0,0412833+	2905	+0,016142−	58	+0,008564−	44	0,710
0,291	003168	1676	201430	408	2425	16	3779	8	0,709	0,291	680072	7085	409928	2915	6084	59	8520	45	0,709
0,292	+0,0001492+	1670	201838	406	2441	16	3787	9	0,708	0,292	672987	7075	407013	2925	6025	59	8475	44	0,708
0,293	−0,0000178−	1662	202244	402	2457	16	3796	8	0,707	0,293	665912	7065	404088	2935	5966	59	8431	45	0,707
0,294	001840	1656	202646	400	2473	16	3804	8	0,706	0,294	658847	7055	401153	2945	5907	59	8386	45	0,706
0,295	003496	1648	203046	397	2489	16	3812	9	0,705	0,295	651792	7045	398208	2955	5848	59	8341	45	0,705
0,296	005144	1641	203443	394	2505	15	3821	8	0,704	0,296	644747	7035	395253	2965	5789	59	8296	45	0,704
0,297	006785	1634	203837	390	2520	16	3829	8	0,703	0,297	637712	7025	392288	2975	5730	60	8251	45	0,703
0,298	008419	1627	204227	388	2536	16	3837	8	0,702	0,298	630687	7015	389313	2985	5670	59	8206	45	0,702
0,299	010046	1621	204615	385	2552	15	3845	9	0,701	0,299	623672	7005	386328	2995	5611	60	8161	46	0,701
0,300	−0,0011667−		−0,0205000−		+0,002567+		+0,003854+		0,700	0,300	−0,1616667+		−0,0383333+		+0,015551−		+0,008115−		0,700

Table VI (cont'd)

n	''E_0		''E_1		''E_0''		''E_1''		m	n	'E_0		'E_1		'E_0''		'E_1''		m
0.300	−0.0011667−	1613	−0.0205000−	382	+0.002567+	16	+0.003854+	8	0.700	0.300	−0.1616667+	6995	−0.0383333+	3005	+0.015551−	59	+0.008115−	45	0.700
0.301	013280	1606	205382	379	2583	15	3862	8	0.699	0.301	609672	6985	380328	3015	5492	60	8070	46	0.699
0.302	014886	1599	205761	375	2598	16	3870	8	0.698	0.302	602687	6975	377313	3025	5432	59	8024	46	0.698
0.303	016485	1592	206136	373	2614	15	3878	8	0.697	0.303	595712	6965	374288	3035	5373	60	7978	46	0.697
0.304	018077	1586	206509	370	2629	15	3886	8	0.696	0.304	588747	6955	371253	3045	5313	60	7932	46	0.696
0.305	019663	1578	206879	367	2644	15	3894	7	0.695	0.305	581792	6945	368208	3055	5253	60	7886	46	0.695
0.306	021241	1571	207246	363	2659	16	3901	8	0.694	0.306	574847	6935	365153	3065	5193	60	7840	46	0.694
0.307	022812	1565	207609	361	2675	15	3909	8	0.693	0.307	567912	6925	362088	3075	5133	60	7794	47	0.693
0.308	024377	1557	207970	357	2690	15	3917	8	0.692	0.308	560987	6915	359013	3085	5073	60	7747	46	0.692
0.309	025934	1551	208327	355	2705	15	3925	7	0.691	0.309	554072	6905	355928	3095	5013	60	7701	47	0.691
0.310	−0.0027485−	1544	−0.0208682−	351	+0.002720+	15	+0.003932+	8	0.690	0.310	−0.1547167+	6895	−0.0352833+	3105	+0.014953−	61	+0.007654−	47	0.690
0.311	029029	1537	209033	348	2735	15	3940	8	0.689	0.311	540272	6885	349728	3115	4892	60	7607	46	0.689
0.312	030566	1529	209381	345	2750	14	3948	7	0.688	0.312	533387	6875	346613	3125	4832	60	7561	47	0.688
0.313	032095	1524	209726	342	2764	15	3955	8	0.687	0.313	526512	6865	343488	3135	4772	61	7514	48	0.687
0.314	033619	1516	210068	339	2779	15	3963	7	0.686	0.314	519647	6855	340353	3145	4711	61	7466	47	0.686
0.315	035135	1509	210407	336	2794	14	3970	8	0.685	0.315	512792	6845	337208	3155	4650	60	7419	47	0.685
0.316	036644	1503	210743	332	2808	15	3978	7	0.684	0.316	505947	6835	334053	3165	4590	61	7372	48	0.684
0.317	038147	1495	211075	329	2823	14	3985	7	0.683	0.317	499112	6825	330888	3175	4529	61	7324	47	0.683
0.318	039642	1489	211404	326	2837	15	3992	7	0.682	0.318	492287	6815	327713	3185	4468	60	7277	48	0.682
0.319	041131	1482	211730	323	2852	14	3999	8	0.681	0.319	485472	6805	324528	3195	4408	61	7229	48	0.681
0.320	−0.0042613−	1476	−0.0212053−	320	+0.002866+	15	+0.004007+	7	0.680	0.320	−0.1478667+	6795	−0.0321333+	3205	+0.014347−	61	+0.007181−	48	0.680
0.321	044089	1468	212373	317	2881	14	4014	7	0.679	0.321	471872	6785	318128	3215	4286	61	7133	48	0.679
0.322	045557	1462	212690	313	2895	14	4021	7	0.678	0.322	465087	6775	314913	3225	4225	61	7085	48	0.678
0.323	047019	1455	213003	310	2909	14	4028	7	0.677	0.323	458312	6765	311688	3235	4164	62	7037	48	0.677
0.324	048474	1448	213313	307	2923	14	4035	7	0.676	0.324	451547	6755	308453	3245	4102	61	6989	48	0.676
0.325	049922	1441	213620	303	2937	14	4042	7	0.675	0.325	444792	6745	305208	3255	4041	61	6941	49	0.675
0.326	051363	1435	213923	301	2951	14	4049	7	0.674	0.326	438047	6735	301953	3265	3980	61	6892	49	0.674
0.327	052798	1428	214224	297	2965	14	4056	7	0.673	0.327	431312	6725	298688	3275	3919	62	6843	48	0.673
0.328	054226	1421	214521	294	2979	14	4063	6	0.672	0.328	424587	6715	295413	3285	3857	61	6795	49	0.672
0.329	055647	1415	214815	290	2993	14	4069	7	0.671	0.329	417872	6705	292128	3295	3796	62	6746	49	0.671
0.330	−0.0057062−	1407	−0.0215105−	287	+0.003007+	13	+0.004076+	7	0.670	0.330	−0.1411167+	6695	−0.0288833+	3305	+0.013734−	61	+0.006697−	49	0.670
0.331	058469	1402	215392	284	3020	14	4083	6	0.669	0.331	404472	6685	285528	3315	3673	62	6648	49	0.669
0.332	059871	1394	215676	281	3034	14	4089	7	0.668	0.332	397787	6675	282213	3325	3611	62	6599	50	0.668
0.333	061265	1388	215957	277	3048	13	4096	6	0.667	0.333	391112	6665	278888	3335	3549	61	6549	49	0.667
0.334	062653	1381	216234	274	3061	14	4102	7	0.666	0.334	384447	6655	275553	3345	3488	62	6500	50	0.666
0.335	064034	1374	216508	270	3075	13	4109	6	0.665	0.335	377792	6645	272208	3355	3426	62	6450	49	0.665
0.336	065408	1368	216778	267	3088	13	4115	7	0.664	0.336	371147	6635	268853	3365	3364	62	6401	50	0.664
0.337	066776	1361	217045	264	3101	14	4122	6	0.663	0.337	364512	6625	265488	3375	3302	62	6351	50	0.663
0.338	068137	1355	217309	261	3115	13	4128	6	0.662	0.338	357887	6615	262113	3385	3240	62	6301	50	0.662
0.339	069492	1348	217570	257	3128	13	4134	7	0.661	0.339	351272	6605	258728	3395	3178	62	6251	50	0.661
0.340	−0.0070840−	1341	−0.0217827−	253	+0.003141+	13	+0.004141+	6	0.660	0.340	−0.1344667+	6595	−0.0255333+	3405	+0.013116−	62	+0.006201−	50	0.660
0.341	072181	1335	218080	251	3154	13	4147	6	0.659	0.341	338072	6585	251928	3415	3054	62	6151	50	0.659
0.342	073516	1328	218331	246	3167	13	4153	6	0.658	0.342	331487	6575	248513	3425	2992	62	6101	51	0.658
0.343	074844	1322	218577	244	3180	13	4159	6	0.657	0.343	324912	6565	245088	3435	2930	63	6050	50	0.657
0.344	076166	1315	218821	240	3193	13	4165	6	0.656	0.344	318347	6555	241653	3445	2867	62	6000	51	0.656
0.345	077481	1309	219061	236	3206	13	4171	6	0.655	0.345	311792	6545	238208	3455	2805	62	5949	50	0.655
0.346	078790	1302	219297	233	3219	12	4177	6	0.654	0.346	305247	6535	234753	3465	2743	63	5899	51	0.654
0.347	080092	1295	219530	230	3231	13	4183	6	0.653	0.347	298712	6525	231288	3475	2680	62	5848	51	0.653
0.348	081387	1289	219760	226	3244	12	4189	5	0.652	0.348	292187	6515	227813	3485	2618	63	5797	51	0.652
0.349	082676	1282	219986	222	3256	13	4194	6	0.651	0.349	285672	6505	224328	3495	2555	62	5746	51	0.651
0.350	−0.0083958−		−0.0220208−		+0.003269+		+0.004200+		0.650	0.350	−0.1279167+		−0.0220833+		+0.012493−		+0.005695−		0.650

m	''E_1		''E_0		''E_1''		''E_0''		n	m	'E_1		'E_0		'E_1''		'E_0''		n
0.350	−0.0083958−	1276	−0.0220208−	219	+0.003269+	12	+0.004200+	6	0.650	0.350	−0.1279167+	6495	−0.0220833+	3505	+0.012493−	63	+0.005695−	52	0.650
0.351	085234	1270	220427	216	3281	13	4206	5	0.649	0.351	272672	6485	217328	3515	2430	62	5643	51	0.649
0.352	086504	1263	220643	212	3294	12	4211	6	0.648	0.352	266187	6475	213813	3525	2368	63	5592	51	0.648
0.353	087767	1256	220855	209	3306	12	4217	5	0.647	0.353	259712	6465	210288	3535	2305	63	5541	52	0.647
0.354	089023	1250	221064	205	3318	13	4222	6	0.646	0.354	253247	6455	206753	3545	2242	63	5489	52	0.646
0.355	090273	1244	221269	201	3331	12	4228	5	0.645	0.355	246792	6445	203208	3555	2179	62	5437	51	0.645
0.356	091517	1237	221470	198	3343	12	4233	6	0.644	0.356	240347	6435	199653	3565	2117	63	5386	52	0.644
0.357	092754	1231	221668	194	3355	12	4239	5	0.643	0.357	233912	6425	196088	3575	2054	63	5334	52	0.643
0.358	093985	1224	221862	191	3367	12	4244	5	0.642	0.358	227487	6415	192513	3585	1991	63	5282	52	0.642
0.359	095209	1218	222053	187	3379	12	4249	5	0.641	0.359	221072	6405	188928	3595	1928	63	5230	52	0.641
0.360	−0.0096427−	1211	−0.0222240−	184	+0.003391+	12	+0.004254+	6	0.640	0.360	−0.1214667+	6395	−0.0185333+	3605	+0.011865−	63	+0.005178−	53	0.640
0.361	097638	1205	222424	179	3403	11	4260	5	0.639	0.361	208272	6385	181728	3615	1802	63	5125	52	0.639
0.362	098843	1199	222603	177	3414	12	4265	5	0.638	0.362	201887	6375	178113	3625	1739	63	5073	53	0.638
0.363	100042	1192	222780	172	3426	12	4270	5	0.637	0.363	195512	6365	174488	3635	1676	63	5020	52	0.637
0.364	101234	1186	222952	169	3438	11	4275	5	0.636	0.364	189147	6355	170853	3645	1613	63	4968	53	0.636
0.365	102420	1180	223121	166	3449	12	4280	5	0.635	0.365	182792	6345	167208	3655	1550	63	4915	53	0.635
0.366	103600	1173	223287	162	3461	11	4285	4	0.634	0.366	176447	6335	163553	3665	1487	64	4862	52	0.634
0.367	104773	1167	223449	158	3472	12	4289	5	0.633	0.367	170112	6325	159888	3675	1423	63	4810	53	0.633
0.368	105940	1161	223607	154	3484	11	4294	5	0.632	0.368	163787	6315	156213	3685	1360	63	4757	53	0.632
0.369	107101	1154	223761	151	3495	11	4299	5	0.631	0.369	157472	6305	152528	3695	1297	64	4704	54	0.631
0.370	−0.0108255−	1148	−0.0223912−	147	+0.003506+	11	+0.004304+	4	0.630	0.370	−0.1151167+	6295	−0.0148833+	3705	+0.011233−	63	+0.004650−	53	0.630
0.371	109403	1142	224059	143	3517	12	4308	5	0.629	0.371	144872	6285	145128	3715	1170	63	4597	53	0.629
0.372	110545	1135	224202	139	3529	11	4313	4	0.628	0.372	138587	6275	141413	3725	1107	64	4544	54	0.628
0.373	111680	1129	224341	136	3540	11	4317	5	0.627	0.373	132312	6265	137688	3735	1043	63	4490	53	0.627
0.374	112809	1123	224477	132	3551	11	4322	4	0.626	0.374	126047	6255	133953	3745	0980	64	4437	54	0.626
0.375	113932	1117	224609	129	3562	11	4326	4	0.625	0.375	119792	6245	130208	3755	0916	63	4383	54	0.625
0.376	115049	1110	224738	124	3573	10	4330	5	0.624	0.376	113547	6235	126453	3765	0853	64	4329	54	0.624
0.377	116159	1105	224862	121	3583	11	4335	4	0.623	0.377	107312	6225	122688	3775	0789	63	4275	54	0.623
0.378	117264	1098	224983	117	3594	11	4339	4	0.622	0.378	101087	6215	118913	3785	0726	64	4221	54	0.622
0.379	118362	1091	225100	113	3605	10	4343	4	0.621	0.379	094872	6205	115128	3795	0662	63	4167	54	0.621
0.380	−0.0119453−	1086	−0.0225213−	110	+0.003615+	11	+0.004347+	4	0.620	0.380	−0.1088667+	6195	−0.0111333+	3805	+0.010599−	64	+0.004113−	54	0.620
0.381	120539	1079	225323	105	3626	11	4351	4	0.619	0.381	082472	6185	107528	3815	0535	64	4059	54	0.619
0.382	121618	1073	225428	102	3637	10	4355	4	0.618	0.382	076287	6175	103713	3825	0471	63	4005	55	0.618
0.383	122691	1068	225530	98	3647	10	4359	4	0.617	0.383	070112	6165	099888	3835	0408	64	3950	54	0.617
0.384	123759	1060	225628	94	3657	11	4363	4	0.616	0.384	063947	6155	096053	3845	0344	64	3896	55	0.616
0.385	124819	1055	225722	91	3668	10	4367	4	0.615	0.385	057792	6145	092208	3855	0280	63	3841	55	0.615
0.386	125874	1049	225813	86	3678	10	4371	4	0.614	0.386	051647	6135	088353	3865	0217	64	3786	54	0.614
0.387	126923	1042	225899	83	3688	10	4375	4	0.613	0.387	045512	6125	084488	3875	0153	64	3732	55	0.613
0.388	127965	1036	225982	78	3698	10	4379	3	0.612	0.388	039387	6115	080613	3885	0089	64	3677	55	0.612
0.389	129001	1031	226060	75	3708	10	4382	4	0.611	0.389	033272	6105	076728	3895	+0.010025	64	3622	55	0.611
0.390	−0.0130032−	1024	−0.0226135−	71	+0.003718+	10	+0.004386+	3	0.610	0.390	−0.1027167+	6095	−0.0072833+	3905	+0.009961−	63	+0.003567−	55	0.610
0.391	131056	1018	226206	67	3728	10	4389	4	0.609	0.391	021072	6085	068928	3915	9898	64	3512	56	0.609
0.392	132074	1012	226273	63	3738	10	4393	3	0.608	0.392	014987	6075	065013	3925	9834	64	3456	55	0.608
0.393	133086	1006	226336	59	3748	10	4396	4	0.607	0.393	008912	6065	061088	3935	9770	64	3401	55	0.607
0.394	134092	999	226395	55	3758	9	4400	3	0.606	0.394	−0.1002847	6055	057153	3945	9706	64	3346	56	0.606
0.395	135091	994	226450	51	3767	10	4403	3	0.605	0.395	−0.0996792	6045	053208	3955	9642	64	3290	56	0.605
0.396	136085	988	226501	48	3777	9	4406	3	0.604	0.396	990747	6035	049253	3965	9578	64	3234	55	0.604
0.397	137073	982	226549	43	3786	10	4409	4	0.603	0.397	984712	6025	045288	3975	9514	64	3179	56	0.603
0.398	138055	975	226592	39	3796	9	4413	3	0.602	0.398	978687	6015	041313	3985	9450	64	3123	56	0.602
0.399	139030	970	226631	36	3805	10	4416	3	0.601	0.399	972672	6005	037328	3995	9386	64	3067	56	0.601
0.400	−0.0140000−		−0.0226667−		+0.003815+		+0.004419+		0.600	0.400	−0.0966667+		−0.0033333+		+0.009322−		+0.003011−		0.600

Table VI (cont'd)

n	"E_0		"E_1		"E_0''		"E_1''		m	n	'E_0		'E_1		'E_0''		'E_1''		m
0.400	−0.0140000−	964	−0.0226667−	31	+0.003815+	9	+0.004419+	3	0.600	0.400	−0.0966667+	5995	−0.0033333+	4005	+0.009322−	64	+0.003011−	56	0.600
0.401	140964	957	226698	27	3824	9	4422	3	0.599	0.401	960672	5985	029328	4015	9258	64	2955	56	0.599
0.402	141921	952	226725	24	3833	9	4425	2	0.598	0.402	954687	5975	025313	4025	9194	64	2899	56	0.598
0.403	142873	946	226749	19	3842	9	4427	3	0.597	0.403	948712	5965	021288	4035	9130	64	2843	57	0.597
0.404	143819	940	226768	15	3851	9	4430	3	0.596	0.404	942747	5955	017253	4045	9066	64	2786	56	0.596
0.405	144759	933	226783	11	3860	9	4433	3	0.595	0.405	936792	5945	013208	4055	9002	64	2730	56	0.595
0.406	145692	928	226794	7	3869	9	4436	2	0.594	0.406	930847	5935	009153	4065	8938	64	2674	57	0.594
0.407	146620	922	226801	3	3878	9	4438	3	0.593	0.407	924912	5925	005088	4075	8874	64	2617	57	0.593
0.408	147542	916	226804	1	3887	9	4441	2	0.592	0.408	918987	5915	−0.0001013+	4085	8810	64	2560	56	0.592
0.409	148458	910	226803	5	3896	9	4443	3	0.591	0.409	913072	5905	+0.0003072−	4095	8746	64	2504	57	0.591
0.410	−0.0149368−	905	−0.0226798−	9	+0.003905+	8	+0.004446+	2	0.590	0.410	−0.0907167+	5895	+0.0007167−	4105	+0.008682−	64	+0.002447−	57	0.590
0.411	150273	898	226789	13	3913	9	4448	3	0.589	0.411	901272	5885	011272	4115	8618	65	2390	57	0.589
0.412	151171	892	226776	18	3922	8	4451	2	0.588	0.412	895387	5875	015387	4125	8553	64	2333	57	0.588
0.413	152063	887	226758	21	3930	9	4453	2	0.587	0.413	889512	5865	019512	4135	8489	64	2276	57	0.587
0.414	152950	881	226737	26	3939	8	4455	2	0.586	0.414	883647	5855	023647	4145	8425	64	2219	57	0.586
0.415	153831	874	226711	30	3947	9	4457	3	0.585	0.415	877792	5845	027792	4155	8361	64	2162	58	0.585
0.416	154705	870	226681	34	3956	8	4460	2	0.584	0.416	871947	5835	031947	4165	8297	64	2104	57	0.584
0.417	155575	863	226647	38	3964	8	4462	2	0.583	0.417	866112	5825	036112	4175	8233	64	2047	58	0.583
0.418	156438	857	226609	42	3972	8	4464	2	0.582	0.418	860287	5815	040287	4185	8169	65	1989	57	0.582
0.419	157295	852	226567	47	3980	8	4466	2	0.581	0.419	854472	5805	044472	4195	8104	64	1932	58	0.581
0.420	−0.0158147−	845	−0.0226520−	51	+0.003988+	8	+0.004468+	1	0.580	0.420	−0.0848667+	5795	+0.0048667−	4205	+0.008040−	64	+0.001874−	57	0.580
0.421	158992	840	226469	55	3996	8	4469	2	0.579	0.421	842872	5785	052872	4215	7976	64	1817	58	0.579
0.422	159832	835	226414	59	4004	8	4471	2	0.578	0.422	837087	5775	057087	4225	7912	64	1759	58	0.578
0.423	160667	828	226355	63	4012	8	4473	2	0.577	0.423	831312	5765	061312	4235	7848	64	1701	58	0.577
0.424	161495	823	226292	68	4020	8	4475	1	0.576	0.424	825547	5755	065547	4245	7784	64	1643	58	0.576
0.425	162318	817	226224	72	4028	7	4476	2	0.575	0.425	819792	5745	069792	4255	7720	65	1585	58	0.575
0.426	163135	811	226152	76	4035	8	4478	1	0.574	0.426	814047	5735	074047	4265	7655	64	1527	58	0.574
0.427	163946	805	226076	81	4043	8	4479	2	0.573	0.427	808312	5725	078312	4275	7591	64	1469	58	0.573
0.428	164751	800	225995	84	4051	7	4481	1	0.572	0.428	802587	5715	082587	4285	7527	64	1411	59	0.572
0.429	165551	794	225911	89	4058	7	4482	1	0.571	0.429	796872	5705	086872	4295	7463	64	1352	58	0.571
0.430	−0.0166345−	788	−0.0225822−	94	+0.004065+	8	+0.004483+	2	0.570	0.430	−0.0791167+	5695	+0.0091167−	4305	+0.007399−	64	+0.001294−	59	0.570
0.431	167133	783	225728	97	4073	7	4485	1	0.569	0.431	785472	5685	095472	4315	7335	64	1235	58	0.569
0.432	167916	777	225631	102	4080	7	4486	1	0.568	0.432	779787	5675	099787	4325	7271	64	1177	59	0.568
0.433	168693	771	225529	107	4087	8	4487	1	0.567	0.433	774112	5665	104112	4335	7207	65	1118	58	0.567
0.434	169464	766	225422	110	4095	7	4488	1	0.566	0.434	768447	5655	108447	4345	7142	64	1060	59	0.566
0.435	170230	760	225312	115	4102	7	4489	1	0.565	0.435	762792	5645	112792	4355	7078	64	1001	59	0.565
0.436	170990	754	225197	119	4109	7	4490	1	0.564	0.436	757147	5635	117147	4365	7014	64	0942	59	0.564
0.437	171744	749	225078	124	4116	7	4491	1	0.563	0.437	751512	5625	121512	4375	6950	64	0883	59	0.563
0.438	172493	743	224954	128	4123	6	4492	1	0.562	0.438	745887	5615	125887	4385	6886	64	0824	59	0.562
0.439	173236	737	224826	133	4129	7	4493	0	0.561	0.439	740272	5605	130272	4395	6822	64	0765	59	0.561
0.440	−0.0173973−	732	−0.0224693−	137	+0.004136+	7	+0.004493+	1	0.560	0.440	−0.0734667+	5595	+0.0134667−	4405	+0.006758−	64	+0.000706−	59	0.560
0.441	174705	726	224556	141	4143	7	4494	1	0.559	0.441	729072	5585	139072	4415	6694	64	0647	59	0.559
0.442	175431	721	224415	146	4150	6	4495	0	0.558	0.442	723487	5575	143487	4425	6630	64	0588	60	0.558
0.443	176152	715	224269	150	4156	7	4495	1	0.557	0.443	717912	5565	147912	4435	6566	64	0528	59	0.557
0.444	176867	710	224119	154	4163	6	4496	0	0.556	0.444	712347	5555	152347	4445	6502	64	0469	59	0.556
0.445	177577	704	223965	159	4169	7	4496	1	0.555	0.445	706792	5545	156792	4455	6438	64	0410	60	0.555
0.446	178281	698	223806	164	4176	6	4497	0	0.554	0.446	701247	5535	161247	4465	6374	64	0350	59	0.554
0.447	178979	693	223642	168	4182	6	4497	0	0.553	0.447	695712	5525	165712	4475	6310	64	0291	60	0.553
0.448	179672	688	223474	172	4188	7	4497	0	0.552	0.448	690187	5515	170187	4485	6246	64	0231	60	0.552
0.449	180360	682	223302	177	4195	6	4497	1	0.551	0.449	684672	5505	174672	4495	6182	64	0171	60	0.551
0.450	−0.0181042−		−0.0223125−		+0.004201+		+0.004498+		0.550	0.450	−0.0679167+		+0.0179167−		+0.006118−		+0.000111−		0.550

m	$''E_1$		$''E_0$		$''E_1''$		$''E_0''$		n	m	$'E_1$		$'E_0$		$'E_1''$		$'E_0''$		n
0.450	−0.0181042−	676	−0.0223125−	181	+0.004201+	6	+0.004498+	0	0.550	0.450	−0.0679167+	5495	+0.0179167−	4505	+0.006118−	64	+0.000111−	59	0.550
0.451	181718	671	222944	186	4207	6	4498	0	0.549	0.451	673672	5485	183672	4515	6054	64	+0.000052−	60	0.549
0.452	182389	665	222758	191	4213	6	4498	0	0.548	0.452	668187	5475	188187	4525	5990	64	−0.000008+	60	0.548
0.453	183054	660	222567	195	4219	6	4498	1	0.547	0.453	662712	5465	192712	4535	5926	64	0068	60	0.547
0.454	183714	655	222372	199	4225	5	4497	0	0.546	0.454	657247	5455	197247	4545	5862	64	0128	60	0.546
0.455	184369	649	222173	204	4230	6	4497	0	0.545	0.455	651792	5445	201792	4555	5798	64	0188	61	0.545
0.456	185018	644	221969	209	4236	6	4497	0	0.544	0.456	646347	5435	206347	4565	5734	63	0249	60	0.544
0.457	185662	638	221760	213	4242	6	4497	1	0.543	0.457	640912	5425	210912	4575	5671	64	0309	60	0.543
0.458	186300	633	221547	218	4248	5	4496	0	0.542	0.458	635487	5415	215487	4585	5607	64	0369	61	0.542
0.459	186933	627	221329	222	4253	6	4496	0	0.541	0.459	630072	5405	220072	4595	5543	64	0430	60	0.541
0.460	−0.0187560−	622	−0.0221107−	227	+0.004259+	5	+0.004496+	1	0.540	0.460	−0.0624667+	5395	+0.0224667−	4605	+0.005479−	63	−0.000490+	60	0.540
0.461	188182	617	220880	232	4264	5	4495	0	0.539	0.461	619272	5385	229272	4615	5416	64	0550	61	0.539
0.462	188799	611	220648	236	4269	6	4495	1	0.538	0.462	613887	5375	233887	4625	5352	64	0611	61	0.538
0.463	189410	606	220412	241	4275	5	4494	1	0.537	0.463	608512	5365	238512	4635	5288	64	0672	60	0.537
0.464	190016	600	220171	245	4280	5	4493	1	0.536	0.464	603147	5355	243147	4645	5224	63	0732	61	0.536
0.465	190616	595	219926	250	4285	5	4492	0	0.535	0.465	597792	5345	247792	4655	5161	64	0793	61	0.535
0.466	191211	590	219671	255	4290	5	4492	1	0.534	0.466	592447	5335	252447	4665	5097	63	0854	61	0.534
0.467	191801	584	219421	260	4295	5	4491	1	0.533	0.467	587112	5325	257112	4675	5034	64	0915	60	0.533
0.468	192385	580	219161	264	4300	5	4490	1	0.532	0.468	581787	5315	261787	4685	4970	64	0975	61	0.532
0.469	192965	573	218897	269	4305	5	4489	1	0.531	0.469	576472	5305	266472	4695	4906	63	1036	61	0.531
0.470	−0.0193538−	569	−0.0218628−	273	+0.004310+	5	+0.004488+	1	0.530	0.470	−0.0571167+	5295	+0.0271167−	4705	+0.004843−	64	−0.001097+	61	0.530
0.471	194107	563	218355	278	4315	5	4487	2	0.529	0.471	565872	5285	275872	4715	4779	63	1158	62	0.529
0.472	194670	558	218077	283	4320	4	4485	1	0.528	0.472	560587	5275	280587	4725	4716	64	1220	61	0.528
0.473	195228	553	217794	288	4324	5	4484	1	0.527	0.473	555312	5265	285312	4735	4652	63	1281	61	0.527
0.474	195781	547	217506	292	4329	5	4483	2	0.526	0.474	550047	5255	290047	4745	4589	63	1342	61	0.526
0.475	196328	542	217214	298	4334	4	4481	1	0.525	0.475	544792	5245	294792	4755	4526	64	1403	62	0.525
0.476	196870	537	216916	302	4338	5	4480	1	0.524	0.476	539547	5235	299547	4765	4462	63	1465	61	0.524
0.477	197407	532	216614	306	4343	4	4479	2	0.523	0.477	534312	5225	304312	4775	4399	63	1526	61	0.523
0.478	197939	526	216308	312	4347	4	4477	2	0.522	0.478	529087	5215	309087	4785	4336	64	1587	62	0.522
0.479	198465	522	215996	316	4351	5	4475	1	0.521	0.479	523872	5205	313872	4795	4272	63	1649	61	0.521
0.480	−0.0198987−	516	−0.0215680−	321	+0.004356+	4	+0.004474+	2	0.520	0.480	−0.0518667+	5195	+0.0318667−	4805	+0.004209−	63	−0.001710+	62	0.520
0.481	199503	511	215359	326	4360	4	4472	2	0.519	0.481	513472	5185	323472	4815	4146	63	1772	62	0.519
0.482	200014	505	215033	331	4364	4	4470	2	0.518	0.482	508287	5175	328287	4825	4083	64	1834	61	0.518
0.483	200519	501	214702	335	4368	4	4468	2	0.517	0.483	503112	5165	333112	4835	4019	63	1895	62	0.517
0.484	201020	495	214367	341	4372	4	4466	2	0.516	0.484	497947	5155	337947	4845	3956	63	1957	62	0.516
0.485	201515	490	214026	345	4376	4	4464	2	0.515	0.485	492792	5145	342792	4855	3893	63	2019	62	0.515
0.486	202005	486	213681	350	4380	3	4462	2	0.514	0.486	487647	5135	347647	4865	3830	63	2081	62	0.514
0.487	202491	479	213331	355	4383	4	4460	2	0.513	0.487	482512	5125	352512	4875	3767	63	2143	62	0.513
0.488	202970	475	212976	360	4387	4	4458	2	0.512	0.488	477387	5115	357387	4885	3704	63	2205	62	0.512
0.489	203445	470	212616	364	4391	3	4456	3	0.511	0.489	472272	5105	362272	4895	3641	63	2267	62	0.511
0.490	−0.0203915−	465	−0.0212252−	370	+0.004394+	4	+0.004453+	2	0.510	0.490	−0.0467167+	5095	+0.0367167−	4905	+0.003578−	62	−0.002329+	62	0.510
0.491	204380	459	211882	374	4398	3	4451	2	0.509	0.491	462072	5085	372072	4915	3516	63	2391	62	0.509
0.492	204839	455	211508	380	4401	4	4449	3	0.508	0.492	456987	5075	376987	4925	3453	63	2453	62	0.508
0.493	205294	449	211128	384	4405	3	4446	2	0.507	0.493	451912	5065	381912	4935	3390	63	2515	62	0.507
0.494	205743	444	210744	390	4408	4	4444	3	0.506	0.494	446847	5055	386847	4945	3327	63	2577	62	0.506
0.495	206187	440	210354	394	4412	3	4441	3	0.505	0.495	441792	5045	391792	4955	3264	62	2639	63	0.505
0.496	206627	434	209960	399	4415	3	4438	2	0.504	0.496	436747	5035	396747	4965	3202	63	2702	62	0.504
0.497	207061	429	209561	404	4418	3	4436	3	0.503	0.497	431712	5025	401712	4975	3139	63	2764	62	0.503
0.498	207490	424	209157	409	4421	3	4433	3	0.502	0.498	426687	5015	406687	4985	3076	62	2826	63	0.502
0.499	207914	419	208748	415	4424	3	4430	3	0.501	0.499	421672	5005	411672	4995	3014	63	2889	62	0.501
0.500	−0.0208333−		−0.0208333−		+0.004427+		+0.004427+		0.500	0.500	−0.0416667+		+0.0416667−		+0.002951−		−0.002951+		0.500

Table VI (cont'd)

n	$''E_0^{iv}$	$''E_1^{iv}$	m	n	$''E_0^{vi}$	$''E_1^{vi}$	m
0.0	+0.00051+	0.00000	1.0	0.0	-0.00008-	0.00000	1.0
	44	31			9	7	
0.1	+0.00007+	-0.00031-	0.9	0.1	+0.00001+	+0.00007+	0.9
	39	28			8	6	
0.2	-0.00032-	-0.00059-	0.8	0.2	+0.00009+	+0.00013+	0.8
	32	21			6	5	
0.3	-0.00064-	-0.00080-	0.7	0.3	+0.00015+	+0.00018+	0.7
	21	13			4	2	
0.4	-0.00085-	-0.00093-	0.6	0.4	+0.00019+	+0.00020+	0.6
	10	2			2	1	
0.5	-0.00095-	-0.00095-	0.5	0.5	+0.00021+	+0.00021+	0.5
m	$''E_1^{iv}$	$''E_0^{iv}$	n	m	$''E_1^{vi}$	$''E_0^{vi}$	n

n	$'E_0^{iv}$	$'E_1^{iv}$	m	n	$'E_0^{vi}$	$'E_1^{vi}$	m
0.0	-0.00448+	-0.00316+	1.0	0.0	+0.00089-	+0.00069-	1.0
	23	17			4	4	
0.1	-0.00425+	-0.00299+	0.9	0.1	+0.00085-	+0.00065-	0.9
	64	48			13	10	
0.2	-0.00361+	-0.00251+	0.8	0.2	+0.00072-	+0.00055-	0.8
	94	77			19	16	
0.3	-0.00267+	-0.00174+	0.7	0.3	+0.00053-	+0.00039-	0.7
	111	99			23	19	
0.4	-0.00156+	-0.00075+	0.6	0.4	+0.00030-	+0.00018-	0.6
	118	113			24	24	
0.5	-0.00038+	+0.00038-	0.5	0.5	+0.00006-	-0.00006+	0.5
m	$'E_1^{iv}$	$'E_0^{iv}$	n	m	$'E_1^{vi}$	$'E_0^{vi}$	n

Table VII

r^2	F_0	D_1	D_2	r^2	F_0	D_1	D_2
4.00	0.1250 0000 3	0.0468 7405	0.0145 2108	5.80	0.0715 9093 2	0.0185 1378	0.0039 3074
4.02	1249 9985 1	468 6041	142 0874	5.85	715 9063 8	185 0223	38 1489
4.05	1249 9914 5	468 3118	139 0474	5.90	697 7855 8	177 3928	37 0340
4.07	1249 9761 8	467 8874	136 0929	5.95	697 7828 6	177 2858	35 9606
4.10	1204 5483 2	440 6799	133 2164	6.00	680 4138 4	170 0939	34 9269
4.12	0.1204 5469 6	0.0440 5578	0.0130 4197	6.05	0.0680 4113 1	0.0169 9946	0.0033 9310
4.15	1204 5406 4	440 2959	127 6960	6.10	663 7511 1	163 2086	32 9714
4.17	1204 5269 4	439 9153	125 0470	6.15	663 7487 6	163 1163	32 0464
4.20	1161 7858 4	414 9158	122 4663	6.20	647 7575 5	156 7073	31 1545
4.22	1161 7846 1	414 8062	119 9555	6.25	647 7553 6	156 6215	30 2943
4.25	0.1161 7789 3	0.0414 5709	0.0117 5088	6.30	0.0632 3961 1	0.0150 5628	0.0029 4643
4.27	1161 7666 2	414 2289	115 1275	6.35	632 3940 8	150 4829	28 6634
4.30	1121 4949 7	391 2123	112 8063	6.40	617 6323 8	144 7504	27 8903
4.32	1121 4938 7	391 1136	110 5465	6.45	617 6304 8	144 6759	27 1437
4.35	1121 4887 5	390 9017	108 3431	6.50	603 4342 9	139 2474	26 4227
4.37	0.1121 4776 5	0.0390 5934	0.0106 1972	6.55	0.0603 4325 2	0.0139 1779	0.0025 7260
4.40	1083 4802 5	369 3620	104 1043	6.60	589 7719 5	134 0328	25 0529
4.42	1083 4792 6	369 2730	102 0655	6.65	589 7703 0	133 9679	24 4022
4.45	1083 4746 4	369 0817	100 0764	6.70	576 6175 0	129 0876	23 7730
4.47	1083 4646 2	368 8033	98 1382	6.75	576 6159 5	129 0269	23 1646
4.50	0.1047 5656 3	0.0349 1759	0.0095 8748	6.80	0.0563 9448 8	0.0124 3942	0.0022 5761
4.53	1047 5636 2	349 0444	93 6780	6.85	563 9434 3	124 3373	22 0066
4.56	1047 5540 8	348 7378	91 1971	6.90	551 7297 3	119 9361	21 4555
4.60	1013 5922 1	330 5078	88 7984	6.95	551 7283 8	119 8829	20 9220
4.63	1013 5903 8	330 3885	86 8073	7.00	539 9492 8	115 6986	20 4054
4.66	0.1013 5817 4	0.0330 1104	0.0084 5570	7.05	0.0539 9480 1	0.0115 6487	0.0019 9051
4.70	981 4162 1	313 2076	82 3797	7.10	528 5821 6	111 6678	19 4205
4.73	981 4145 5	313 0993	80 5711	7.15	528 5809 7	111 6209	18 9509
4.76	981 4066 9	312 8465	78 5257	7.20	517 6083 6	107 8308	18 4959
4.80	950 9072 5	297 1491	76 5454	7.25	517 6072 4	107 7868	18 0547
4.83	0.0950 9057 4	0.0297 0505	0.0074 8993	7.30	0.0507 0090 8	0.0104 1760	0.0017 6271
4.86	950 8985 8	296 8203	73 0365	7.35	507 0080 3	104 1346	17 2123
4.90	921 9468 7	282 2200	71 2317	7.40	496 7666 7	100 6922	16 8100
4.93	921 9454 9	282 1301	69 7306	7.45	496 7656 8	100 6533	16 4198
4.96	921 9389 6	281 9200	68 0309	7.50	486 8645 3	97 3694	16 0411
5.00	0.0894 4272 2	0.0268 3203	0.0066 3830	7.55	0.0486 8636 0	0.0097 3327	0.0015 6735
5.03	894 4259 7	268 2382	65 0117	7.60	477 2870 4	94 1981	15 3168
5.06	894 4199 9	268 0461	63 4579	7.65	477 2861 6	94 1635	14 9704
5.10	868 2499 2	255 3604	61 9506	7.70	468 0195 0	91 1695	14 6339
5.13	868 2487 7	255 2853	60 6955	7.75	468 0186 7	91 1369	14 3072
5.16	0.0868 2433 0	0.0255 1094	0.0059 2726	7.80	0.0459 0480 3	0.0088 2755	0.0013 9898
5.20	843 3250 1	243 2487	57 5056	7.85	459 0472 5	88 2447	13 6814
5.25	843 3202 3	243 0607	55 6203	7.90	450 3595 7	85 5085	13 3816
5.30	819 5702 7	231 9372	53 8137	7.95	450 3588 3	85 4794	13 0903
5.35	819 5658 7	231 7645	52 0819	8.00	441 9417 7	82 8614	12 8070
5.40	0.0796 9101 6	0.0221 3486	0.0050 4211	8.05	0.0441 9410 8	0.0082 8340	0.0012 5316
5.45	796 9061 1	221 1897	48 8278	8.10	433 7829 8	80 3277	12 2638
5.50	775 2753 4	211 4246	47 2986	8.15	433 7823 2	80 3017	12 0032
5.55	775 2716 1	211 2782	45 8304	8.20	425 8721 7	77 9011	11 7498
5.60	754 6020 2	202 1125	44 4203	8.25	425 8715 4	77 8765	11 5032
5.65	0.0754 5985 8	0.0201 9774	0.0043 0655	8.30	0.0418 1988 7	0.0075 5693	0.0011 1459
5.70	734 8314 3	193 3647	41 7634	8.40	410 7533 1	73 3404	10 6910
5.75	734 8282 5	193 2398	40 5115	8.50	403 5260 9	71 2026	10 2597
5.80	715 9093 2	185 1378	39 3074	8.60	396 5083 4	69 1509	9 8506

Table VII (cont'd)

r^2	F_0	D_1	D_2	r^2	F_0	D_1	D_2
8.60	0.0396 5083 4	0.0069 1509	0.0009 8506	13.90	0.01929 6466 9	0.00208 2265	0.00018 4926
8.70	389 6916 8	67 1811	9 4621	14.00	1909 0088 7	204 5282	18 0360
8.80	383 0681 1	65 2889	9 0932	14.10	1888 7363 2	200 9212	17 5938
8.90	376 6300 8	63 4705	8 7425	14.20	1868 8200 4	197 4026	17 1654
9.00	370 3703 8	61 7223	8 4090	14.30	1849 2513 5	193 9697	16 7504
9.10	0.0364 2821 8	0.0060 0407	0.0008 0917	14.40	0.01830 0218 0	0.00190 6198	0.00016 3482
9.20	358 3589 7	58 4225	7 7897	14.50	1811 1232 1	187 3503	15 9583
9.30	352 5945 5	56 8648	7 5019	14.60	1792 5476 8	184 1588	15 5803
9.40	346 9830 4	55 3646	7 2278	14.70	1774 2875 2	181 0429	15 2137
9.50	341 5188 0	53 9192	6 9663	14.80	1756 3353 0	178 0004	14 8581
9.60	0.0336 1964 9	0.0052 5261	0.0006 7169	14.90	0.01738 6837 7	0.00175 0289	0.00014 5132
9.70	331 0110 0	51 1829	6 4788	15.00	1721 3259 4	172 1264	14 1784
9.80	325 9574 6	49 8873	6 2515	15.10	1704 2550 2	169 2908	13 8536
9.90	321 0312 0	48 6371	6 0344	15.20	1687 4644 1	166 5202	13 5382
10.00	316 2277 9	47 4303	5 8268	15.30	1670 9477 0	163 8127	13 2320
10.10	0.0311 5429 8	0.0046 2651	0.0005 6284	15.40	0.01654 6986 9	0.00161 1664	0.00012 9347
10.20	306 9727 2	45 1395	5 4385	15.50	1638 7113 3	158 5796	12 6459
10.30	302 5131 1	44 0520	5 2568	15.60	1622 9797 7	156 0505	12 3653
10.40	298 1604 5	43 0007	5 0828	15.70	1607 4983 1	153 5776	12 0927
10.50	293 9111 7	41 9842	4 9162	15.80	1592 2614 3	151 1591	11 8278
10.60	0.0289 7618 8	0.0041 0011	0.0004 7565	15.90	0.01577 2637 4	0.00148 7937	0.00011 5703
10.70	285 7093 0	40 0499	4 6034	16.00	1562 5000 2	146 4797	11 3199
10.80	281 7503 2	39 1293	4 4566	16.10	1547 9651 9	144 2158	11 0765
10.90	277 8819 3	38 2380	4 3158	16.20	1533 6543 2	142 0006	10 8397
11.00	274 1012 5	37 3749	4 1806	16.30	1519 5626 1	139 8328	10 6095
11.10	0.0270 4055 4	0.0036 5389	0.0004 0509	16.40	0.01505 6853 8	0.00137 7110	0.00010 3854
11.20	266 7921 3	35 7288	3 9262	16.50	1492 0181 0	135 6340	10 1675
11.30	263 2584 9	34 9436	3 8065	16.60	1478 5563 3	133 6005	9 9553
11.40	259 8021 7	34 1824	3 6914	16.70	1465 2957 9	131 6096	9 7489
11.50	256 4208 2	33 4441	3 5807	16.80	1452 2322 8	129 6599	9 5479
11.60	0.0253 1122 0	0.0032 7281	0.0003 4743	16.90	0.01439 3617 3	0.00127 7504	0.00009 3521
11.70	249 8741 1	32 0332	3 3719	17.00	1426 6801 7	125 8800	9 1616
11.80	246 7044 9	31 3589	3 2734	17.10	1414 1837 5	124 0478	8 9760
11.90	243 6013 1	30 7043	3 1785	17.20	1401 8686 9	122 2526	8 7952
12.00	240 5626 5	30 0686	3 0871	17.30	1389 7313 5	120 4937	8 6190
12.10	0.0237 5866 4	0.0029 4513	0.0002 9991	17.40	0.01377 7681 3	0.00118 7699	0.00008 4474
12.20	234 6714 9	28 8515	2 9143	17.50	1365 9755 8	117 0805	8 2801
12.30	231 8154 6	28 2687	2 8325	17.60	1354 3503 0	115 4246	8 1171
12.40	229 0169 1	27 7022	2 7537	17.70	1342 8889 8	113 8012	7 9581
12.50	226 2742 1	27 1515	2 6777	17.80	1331 5884 1	112 2096	7 8032
12.60	0.0223 5858 2	0.0026 6160	0.0002 6043	17.90	0.01320 4454 4	0.00110 6490	0.00007 6521
12.70	220 9502 5	26 0952	2 5335	18.00	1309 4570 3	109 1187	7 5047
12.80	218 3660 5	25 5885	2 4652	18.10	1298 6201 8	107 6178	7 3610
12.90	215 8318 4	25 0955	2 3992	18.20	1287 9319 8	106 1456	7 2208
13.00	213 3462 7	24 6157	2 3355	18.30	1277 3896 0	104 7015	7 0840
13.10	0.0210 9080 4	0.0024 1486	0.0002 2739	18.40	0.01266 9902 6	0.00103 2848	0.00006 9505
13.20	208 5159 0	23 6939	2 2144	18.50	1256 7312 6	101 8947	6 8203
13.30	206 1686 5	23 2510	2 1569	18.60	1246 6099 7	100 5307	6 6931
13.40	203 8651 0	22 8197	2 1013	18.70	1236 6238 0	99 1921	6 5690
13.50	201 6041 4	22 3995	2 0475	18.80	1226 7702 5	97 8783	6 4478
13.60	0.0199 3846 5	0.0021 9900	0.0001 9955	18.90	0.01217 0468 7	0.00096 5888	0.00006 3295
13.70	197 2056 0	21 5909	1 9451	19.00	1207 4512 6	95 3229	6 2140
13.80	195 0659 5	21 2019	1 8964	19.10	1197 9810 9	94 0802	6 1012
13.90	192 9647 1	20 8227	1 8493	19.20	1188 6340 6	92 8600	5 9910

Table VII (cont'd)

r^2	F_0	D_1	D_2	r^2	F_0	D_1	D_2
19.20	0.01188 6340 6	0.00092 8600	0.00005 9910	27.00	0.00712 7781 3	0.00039 5970	0.00001 8097
19.30	1179 4079 4	91 6618	5 8833	27.20	712 7762 7	39 5787	1 7638
19.40	1170 3005 7	90 4852	5 7781	27.40	712 7691 6	39 5432	1 7193
19.50	1161 3098 1	89 3296	5 6753	27.60	712 7536 2	39 4915	1 6762
19.60	1152 4335 9	88 1946	5 5749	27.80	712 7269 2	39 4248	1 6345
19.70	0.01143 6698 5	0.00087 0796	0.00005 4767	28.00	0.00674 9365 9	0.00036 1558	0.00001 5942
19.80	1135 0166 4	85 9843	5 3807	28.20	674 9349 9	36 1402	1 5551
19.90	1126 4719 9	84 9082	5 2869	28.40	674 9289 0	36 1099	1 5172
20.00	1118 0340 2	83 8508	5 1952	28.60	674 9156 7	36 0659	1 4805
20.10	1109 7008 7	82 8118	5 1055	28.80	674 8929 2	36 0091	1 4450
20.20	0.01101 4707 3	0.00081 7907	0.00005 0178	29.00	0.00640 3287 9	0.00033 1192	0.00001 4105
20.30	1093 3418 1	80 7872	4 9321	29.20	640 3267 5	33 1010	1 3689
20.40	1085 3124 0	79 8008	4 8482	29.50	640 3175 0	33 0626	1 3289
20.50	1077 3807 8	78 8312	4 7661	29.70	640 2976 0	33 0073	1 2904
20.60	1069 5453 0	77 8780	4 6858	30.00	608 5806 5	30 4279	1 2532
20.70	0.01061 8043 4	0.00076 9409	0.00004 6072	30.20	0.00608 5789 3	0.00030 4124	0.00001 2175
20.80	1054 1563 1	76 0195	4 5304	30.50	608 5709 1	30 3792	1 1830
20.90	1046 5996 5	75 1134	4 4551	30.70	608 5537 3	30 3315	1 1498
21.00	1039 1328 5	74 2224	4 3815	31.00	579 3719 8	28 0332	1 1177
21.10	1031 7544 1	73 3461	4 3094	31.20	579 3704 6	28 0197	1 0869
21.20	0.01024 4628 8	0.00072 4843	0.00004 2388	31.50	0.00579 3635 7	0.00027 9911	0.00001 0571
21.30	1017 2568 2	71 6365	4 1698	31.70	579 3486 5	27 9497	1 0283
21.40	1010 1348 5	70 8026	4 1021	32.00	552 4272 1	25 8942	1 0005
21.50	1003 0956 0	69 9822	4 0359	32.20	552 4258 9	25 8825	9738
21.60	996 1377 3	69 1750	3 9710	32.50	552 4199 3	25 8577	9479
21.70	0.00989 2599 2	0.00068 3808	0.00003 9075	32.70	0.00552 4070 1	0.00025 8218	0.00000 9229
21.80	982 4609 1	67 5994	3 8452	33.00	527 5080 8	23 9769	8987
21.90	975 7394 0	66 8303	3 7843	33.20	527 5069 4	23 9667	8753
22.00	969 0941 6	66 0702	3 6952	33.50	527 5017 1	23 9450	8527
22.20	969 0894 9	66 0244	3 5805	33.70	527 4903 7	23 9136	8309
22.40	0.00969 0718 3	0.00065 9364	0.00003 4703	34.00	0.00504 4076 5	0.00022 2527	0.00000 8098
22.60	969 0336 9	65 8095	3 3645	34.20	504 4066 4	22 2437	7893
22.80	968 9686 0	65 6469	3 2628	34.50	504 4020 9	22 2248	7696
23.00	906 5844 2	59 1215	3 1650	34.70	504 3921 5	22 1972	7504
23.20	906 5805 8	59 0839	3 0709	35.00	482 9453 1	20 6964	7282
23.40	0.00906 5660 6	0.00059 0116	0.00002 9804	35.30	0.00482 9433 8	0.00020 6837	0.00000 7069
23.60	906 5347 0	58 9072	2 8933	35.60	482 9341 6	20 6541	6830
23.80	906 4810 5	58 7731	2 8094	36.00	462 9629 8	19 2890	6601
24.00	850 5172 8	53 1543	2 7287	36.30	462 9612 8	19 2778	6413
24.20	850 5141 2	53 1233	2 6509	36.60	462 9531 7	19 2517	6202
24.40	0.00850 5020 9	0.00053 0634	0.00002 5759	37.00	0.00444 3216 2	0.00018 0121	0.00000 6000
24.60	850 4760 4	52 9767	2 5036	37.30	444 3201 0	18 0021	5833
24.80	850 4314 7	52 8653	2 4339	37.60	444 3129 5	17 9791	5647
25.00	800 0000 2	47 9975	2 3667	38.00	426 8985 1	16 8504	5467
25.20	799 9973 7	47 97 6	2 3019	38.30	426 8971 7	16 8416	5320
25.40	0.00799 9873 3	0.00047 9216	0.00002 2393	38.60	0.00426 8907 9	0.00016 8211	0.00000 5154
25.60	799 9655 8	47 8492	2 1789	39.00	410 5850 5	15 7910	4994
25.80	799 9282 9	47 7560	2 1206	39.30	410 5838 5	15 7831	4862
26.00	754 2928 5	43 5148	2 0643	39.60	410 5781 7	15 7648	4714
26.20	754 2906 4	43 4931	2 0098	40.00	395 2847 4	14 8225	4572
26.40	0.00754 2822 0	0.00043 4511	0.00001 9573				
26.60	754 2638 9	43 3901	1 9064				
26.80	754 2324 7	43 3116	1 8573				
27.00	712 7781 3	39 5970	1 8097				

INDEX